Quantum Markov Chains and Position Distributions for States of Boson Systems

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Introduction

The present work focuses on applications of classical probability theory, especially point process theory, to quantum stochastics.
In classical statistical mechanics the state of a particle system can be described by a probability measure on a suitable phase space. Due to this, point process theory obtained wide application in classical statistical mechanics.
In [13, 16] an analogue application to statistical quantum mechanics was introduced. The point process defined there (called position distribution) contains all information about position measurements at the system. Of course for a non-classical system this point process describes only one aspect of its state.
One possible access to quantum mechanics is the algebraic one. The system is modelled by an algebra of observables, the state of the system is a functional on this algebra.
More precisely, the state ω_t of a quantum mechanical system at time $t \in T$ can be described as a linear functional on an algebra $\mathcal{A}$ of observables in the form

$$\omega_t = \omega \circ \tau_t \qquad (t \in T)$$

with an initial state $\omega = \omega_0$ and

$$\tau_t(A) = U_t^* A U_t \qquad (A \in \mathcal{A}).$$

Here $(U_t)_{t\in T}$ is a semigroup of unitary operators describing the time evolution of the system. This evolution may be continuous ($T = [0, \infty)$) or discrete ($T = \mathbb{N}$) in time. If in addition to this evolution there is also a repeated (for $T = \mathbb{N}$) or continuous (for $T = [0, \infty)$) measurement, the assumption concerning the mappings τ_t has to be dropped. To describe the 'perturbed' evolution of the system in general, one will only require $(\tau_t)_{t\in T}$ to be a semigroup of completely positive linear mappings from $\mathcal{A}$ to $\mathcal{A}$.
In the work at hand we consider discrete evolutions und measurements described by generalized beam splitting procedures. These models are generalizations of those given in [2, 38].
Quantum Markov processes as continuous time versions of quantum Markov chains and corresponding convergence statements are for instance treated in [32, 33].

Modelling light beams, one considers states on a quasilocal algebra acting on a symmetric Fock space over some phase space G. The symmetric Fock space describes a bosonic field. The publications [16, 17, 22] revealed a strong connection

between the theory of boson states and classical point process theory using the notion of position distribution. The basis of this connection is a correspondence of Poisson processes and coherent states. In this context, beam splitting procedures correspond to random splittings of point configurations. Systems of finitely many bosons are modelled by normal states, locally finite boson systems are described by locally normal states [14, 15, 17, 22].
The splitting procedure itself corresponds to a completely positive identity preserving map (called transition expectation, [2]) from the double tensor product of the boson algebra into the boson algebra.
In the case of the so-called independent splitting, the procedure is modelled using two complex-valued functions (splitting rates) α and β on the phase space G which satisfy $|\alpha(x)|^2 + |\beta(x)|^2 = 1$ for all $x \in G$, [26]. By interaction with a measurement apparatus, a coherent signal is split into two also coherent signals of lower intensity (One can imagine the apparatus as a half-reflecting mirror.). With probability $|\alpha|^2$ the beam is absorbed (or reflected or destroyed), with probability $|\beta|^2$ it passes through. The beam after the measurement can be split again. By repeating this procedure again and again and considering the sequence of the results of interaction with the measurement apparatus, one gets a quantum Markov chain in the sense of ACCARDI [1, 2, 4]. In [24] there was considered a special class of such quantum Markov chains.
Time evolutions of the quantum system and related invariance questions for independent splittings were discussed in [26, 19].
In this case we will add independent evolutions of the quantum system and the measurement apparatus.
The independent beam splitting model was generalized to dependent splittings in [27].
One aim of this work is to bundle the results of numerous publications on beam splitting models and corresponding quantum Markov chains and to provide a common platform. We consider general interaction procedures with two outputs in every time step n. Because of the history of origins we still call these procedures generalized splittings, although they are more general interactions and not necessarily splittings as described above. As far as possible, we do all calculations for this generalized interaction procedure.
Many of the quoted papers [19, 24, 33] consider only diffuse reference measures ν on the phase space, i.e. $\nu(x) = 0$ for all singletons $x \in G$. [32] does not use this restriction.
In this work we use general locally finite reference measures, atoms are allowed.

For the models described above we will discuss the following questions:

1. Given an initial state ω_0, what does the state ω_n at time n look like? Can it be described in an explicit form?
   What shape does the position distribution of the state ω_n assume?

2. Which states ω are stationary ($\omega = \omega_n$ for all $n \in \mathbb{N}$)? Such states describe equilibrium systems.

3. For which initial states ω do the states ω_n converge in some sense towards a stationary state as n tends to ∞?

Now we want to give the structure of this work and the contents of the chapters.
The first two chapters provide the necessary notions from quantum mechanics and point process theory. Chapter 1 contains the mainly algebraic part coming from quantum mechanics. Basic notions like transition expectation, quantum Markov chain and the quasilocal algebra are introduced. Hereby we follow mainly [2, 3, 8].
In chapter 2 we define the symmetric Fock space in the language of point process theory and prepare technical tools, especially for using also atomic reference measures. We introduce generalized binomial coefficients and prove the corresponding calculation rules in detail to handle multiplicities in the point configurations ([32, 28]). We give also a complete proof of the so-called $\sum\!\!\!\!\!\!\int$ - Lemma for the case of atomic reference measure ([30, 17, 24, 35, 36]).
Chapter 3 describes the generalization of the beam splitting procedures considered in [26, 32, 27, 28]. In section 3.1 we define transition expectations $\mathcal{E}_{U_1,U_2}$ to model a quantum measurement process with additional independent inner evolutions of the quantum system and the measurement apparatus (given by isometric operators U_1 and U_2). We compute the transition expectation $\mathcal{E}^n_{U_1,U_2}$ corresponding to the nth step in a chain of such generalized splitting procedures (Prop. 3.4 and Prop. 3.6). In section 3.2 we consider quantum Markov chains with transition expectation $\mathcal{E}_{U_1,U_2}$. The quantum Markov chain describes the behaviour of the measurement apparatus. We give descriptions of the states $\omega_{n]}^{U_1,U_2}$ up to time n (Prop. 3.10 and Prop. 3.12) and $\omega_n^{U_1,U_2}$ at time n (Prop. 3.13 and Prop. 3.14) of the quantum Markov chain.
In section 3.3 we develop explicit formulae for the corresponding position distributions. Proposition 3.21 characterizes the position distribution at time n using a recursive representation of stochastic kernels.
Finally, in section 3.4 the so-called geometric splitting ([27, 32]) is discussed as an example for a non-independent splitting. We give a detailed description of the splitting function g and develop a recursive formula for the position distribution (Corollary 3.25). Furthermore, we find a special characterization of the position distribution in step 1 (Corollary 3.26).
Chapter 4 includes the application of the results from chapter 3 to the independent beam splitting. For this purpose we condense the results from [19, 26, 33, 32] and give some proofs which were omitted there or only sketched. For instance, we give the proof of a formula for the position distribution, containing a convolutional representation, which was sketched in [32] (Lemma 4.12 and Proposition 4.13). In section 4.3 we summarize some results on invariant normal states from [26] and derive conclusions about the evolution of the measurement apparatus.
In chapter 5 infinite, locally finite systems, i.e. locally normal states on the quasilocal algebra, are considered. On the basis of the results of [26] we discuss locally normal states that are invariant under the beam splitting and the consequences for the behaviour of the measurement process.
For normal and locally normal states only the vacuum state satisfies the invariance equation. That's why in chapter 6 we include the second quantizations of contraction

operators in the splitting procedure to compensate the loss caused by the splitting (as it was proposed in [26]). Section 6.1 contains the definition of the contraction operators (with a representation different from the one in [26]) and describes their properties. In section 6.2 we find a condition for invariance of a locally normal coherent state under independent splitting with contraction (Prop. 6.11) and give three examples for states fulfilling it. Finally, in section 6.3 we search for conditions which ensure convergence to invariant states. We discuss an example analogue to Example 5.1 in [26] and give another example for convergence to a non-vacuum invariant state with a necessary and sufficient condition for this convergence (Prop. 6.21).

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Chapter 1

Notations and Basic Notions

In this section we will define some notions connected with the quantum measurement process. Hereby, we will mainly follow [2] and [3].
The quantum measurement process results from an interaction between the quantum system and the measurement apparatus.
Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras representing the algebra of observables of the quantum system and the measurement apparatus, respectively. The measurement is described by a larger system $\mathcal{C}$ of observables containing embeddings of $\mathcal{A}$ and $\mathcal{B}$. The most common choice for $\mathcal{C}$ is a fixed C^*-product $\mathcal{B} \otimes \mathcal{A}$.
Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{D}$ a von Neumann subalgebra of the algebra $\mathfrak{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Since we will only consider algebras contained in von Neumann algebras $\mathfrak{L}(\mathcal{H})$, there will be a natural choice of the C^*-tensor product (see [8], chapter 2.7.2). By $\mathcal{D}^n := \mathcal{D}^{\otimes n}$, $n \geq 1$, we denote the n-fold tensor product of $\mathcal{D}$. Furthermore, we will assume that all C^*-algebras $\mathcal{D}$ considered in the following possess an identity $\mathbb{1}_{\mathcal{D}}$. We set $\mathbb{1}_{\mathcal{D}}^n := \mathbb{1}_{\mathcal{D}}^{\otimes n}$.
A STATE ω on a C^*-algebra $\mathcal{D}$ is a positive continuous linear functional on $\mathcal{D}$ with $\|\omega\| = 1$. The set of all states on $\mathcal{D}$ is denoted by $\mathcal{S}(\mathcal{D})$.

1.1 Channels, Liftings and Transition Expectations

To describe the measurement process our general goal is to construct a map from the state space of one system to the state space of another system. Such a map is called a CHANNEL. We use the notion of special channels, the so-called LIFTINGS, to characterize the measurement process. A lifting is a channel from the state space of an algebra $\mathcal{A}$ to the state space of the algebra $\mathcal{B} \otimes \mathcal{A}$. By the measurement a state on $\mathcal{A}$ (i.e. the preparation of the quantum system) is transformed into a state of the compound system $\mathcal{B} \otimes \mathcal{A}$, representing the whole system after the measurement.

Definition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A mapping $\mathcal{E}^* : \mathcal{S}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{B} \otimes \mathcal{A})$ is called a LIFTING.*

Two important channels connected with the lifting $\mathcal{E}^*$ are the following:
If the quantum system before the measurement was prepared according to the state ρ, then

$$\begin{aligned} \Lambda^*_{\mathcal{A},\mathcal{E}^*} &: \quad \mathcal{S}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{A}) \\ \Lambda^*_{\mathcal{A},\mathcal{E}^*}\rho(A) &:= \quad (\mathcal{E}^*\rho)(\mathbb{1}_{\mathcal{B}} \otimes A) \qquad (\rho \in \mathcal{S}(\mathcal{A}), A \in \mathcal{A}) \end{aligned} \tag{1.1.1}$$

describes the state of the quantum system after the measurement characterized by $\mathcal{E}^*$ and

$$\begin{aligned} \Lambda^*_{\mathcal{B},\mathcal{E}^*} &: \quad \mathcal{S}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{B}) \\ \Lambda^*_{\mathcal{B},\mathcal{E}^*}\rho(B) &:= \quad (\mathcal{E}^*\rho)(B \otimes \mathbb{1}_{\mathcal{A}}) \qquad (\rho \in \mathcal{S}(\mathcal{A}), B \in \mathcal{B}) \end{aligned} \tag{1.1.2}$$

describes the state of the measurement apparatus after the interaction with the quantum system according to $\mathcal{E}^*$.

Usually liftings are defined by duality from transition expectations.
Let for $n \in \mathbb{N}$

$$n] := \{1, ..., n\}. \tag{1.1.3}$$

Remember that we consider only algebras $\mathcal{A}$, $\mathcal{B}$ equal or contained in von Neumann algebras $\mathfrak{L}(\mathcal{H})$.

Definition 1.2 (CP1). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear mapping $\mathcal{E} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is called TRANSITION EXPECTATION if it is completely positive and identity preserving, i.e. for all $n \in \mathbb{N}$*

$$\sum_{j,k=1}^{n} B_j^* \mathcal{E}(C_j^* C_k) B_k \geq 0 \qquad (B_l \in \mathcal{A},\ C_l \in \mathcal{B} \otimes \mathcal{A},\ l \in n]\ )$$

and

$$\mathcal{E}(\mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{A}}.$$

Transition expectations are used to construct quantum Markov chains as it was shown in [2, 4].
A transition expectation of the form $\mathcal{E} = \mathcal{V}^*(.)\mathcal{V}$ with $\mathcal{V}$ being an isometry is called ISOMETRIC TRANSITION EXPECTATION.
For a given transition expectation $\mathcal{E} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ we get the corresponding lifting $\mathcal{E}^* : \mathcal{S}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{B} \otimes \mathcal{A})$ by

$$[\mathcal{E}^*\rho](C) := \rho(\mathcal{E}(C)) \qquad (C \in \mathcal{B} \otimes \mathcal{A}, \rho \in \mathcal{S}(\mathcal{A})\ ).$$

Because $\mathcal{E}$ is linear, completely positive and identity preserving, $(\mathcal{E}^*\rho)$ is a positive linear functional with $(\mathcal{E}^*\rho)(\mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}}) = 1$.
Analogously, we get the linear maps
$\Lambda_{\mathcal{A},\mathcal{E}} : \mathcal{A} \longrightarrow \mathcal{A}$ with

$$\Lambda_{\mathcal{A},\mathcal{E}}(A) := \mathcal{E}(\mathbb{1}_{\mathcal{B}} \otimes A) \qquad (A \in \mathcal{A})$$

and $\Lambda_{\mathcal{B},\mathcal{E}} : \mathcal{B} \longrightarrow \mathcal{A}$ with

$$\Lambda_{\mathcal{B},\mathcal{E}}(B) := \mathcal{E}(B \otimes \mathbb{1}_{\mathcal{A}}) \qquad (B \in \mathcal{B}).$$

$\Lambda_{\mathcal{A},\mathcal{E}}$ describes the evolution of the quantum system.
$\Lambda_{\mathcal{B},\mathcal{E}}$ describes the evolution of the measurement apparatus.

Especially important for us will be isometric transition expectations.

Lemma 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\mathcal{E} : \mathcal{B}\otimes\mathcal{A} \longrightarrow \mathcal{A}$ an identity preserving linear map of the form $\mathcal{E} = \mathcal{V}^*(.)\mathcal{V}$ with an isometric operator $\mathcal{V} : \mathcal{A} \longrightarrow \mathcal{B} \otimes \mathcal{A}$. Then $\mathcal{E}$ is a transition expectation.*

PROOF. For all $B_l \in \mathcal{A}$, $C_l \in \mathcal{B} \otimes \mathcal{A}$, $l \in n]$, $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{j,k=1}^{n} B_j^* \mathcal{E}(C_j^* C_k) B_k &= \sum_{j,k=1}^{n} B_j^* \mathcal{V}^*(C_j^* C_k)\mathcal{V} B_k = \sum_{j,k=1}^{n} (C_j \mathcal{V} B_j)^*(C_k \mathcal{V} B_k) \\ &= \sum_{j=1}^{n} (C_j \mathcal{V} B_j)^* \sum_{k=1}^{n} (C_k \mathcal{V} B_k) \geq 0.\end{aligned}$$

Hence $\mathcal{E}$ is completely positive and, because it is also identity preserving, a transition expectation. □

One can also include additional transformations of the quantum system and/or the measurement equipment:

Lemma 1.4. *Let $\mathcal{E} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be given as in Lemma 1.3. Furthermore, let $\mathcal{V}_1 \in \mathcal{B}$, $\mathcal{V}_2 \in \mathcal{A}$ be isometric operators and $\mathcal{K}_1 = \mathcal{V}_1^*(.)\mathcal{V}_1$, $\mathcal{K}_2 = \mathcal{V}_2^*(.)\mathcal{V}_2$ linear, identity preserving maps.*
Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are transition expectations and $\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ with

$$\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}(B \otimes A) := \mathcal{E}(\mathcal{K}_1(B) \otimes \mathcal{K}_2(A)) = \mathcal{E}(\mathcal{V}_1^* B \mathcal{V}_1 \otimes \mathcal{V}_2^* A \mathcal{V}_2) \qquad (A \in \mathcal{A}\ ,\ B \in \mathcal{B})$$

is a transition expectation.

PROOF. The fact that $\mathcal{K}_1$ and $\mathcal{K}_2$ are transition expectations follows immediately from Lemma 1.3.
Let $A \in \mathcal{A}$, $B \in \mathcal{B}$. Then

$$\begin{aligned}\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}(B \otimes A) &= \mathcal{E}(\mathcal{V}_1^* B \mathcal{V}_1 \otimes \mathcal{V}_2^* A \mathcal{V}_2) = \mathcal{V}^*(\mathcal{V}_1^* B \mathcal{V}_1 \otimes \mathcal{V}_2^* A \mathcal{V}_2)\mathcal{V} \\ &= \mathcal{V}^*(\mathcal{V}_1^* \otimes \mathbb{1}_{\mathcal{A}})(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2^*)(B \otimes A)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} \\ &= \Big((\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}\Big)^* (B \otimes A)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} \\ &= \widehat{\mathcal{V}}^*(B \otimes A)\widehat{\mathcal{V}}\end{aligned}$$

with $\widehat{\mathcal{V}} := (\mathbb{1} \otimes \mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1})\mathcal{V}$. Hence, $\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2} = \widehat{\mathcal{V}}^*(\cdot)\widehat{\mathcal{V}}$.
To apply Lemma 1.3 we still have to show that $\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}$ is identity preserving and that $\widehat{\mathcal{V}}$ is an isometry. Since $\mathcal{V}$, $\mathcal{V}_1$ and $\mathcal{V}_2$ are isometries we have

$$\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}(\mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}}) = \mathcal{V}^*(\mathcal{V}_1^* \mathbb{1}_{\mathcal{B}} \mathcal{V}_1 \otimes \mathcal{V}_2^* \mathbb{1}_{\mathcal{A}} \mathcal{V}_2)\mathcal{V} = \mathcal{V}^*(\mathcal{V}_1^* \mathcal{V}_1 \otimes \mathcal{V}_2^* \mathcal{V}_2)\mathcal{V} = \mathcal{V}^*\mathcal{V} = \mathbb{1}_{\mathcal{A}}.$$

and

$$\begin{aligned}
\widehat{\mathcal{V}}^*\widehat{\mathcal{V}} &= \mathcal{V}^*(\mathcal{V}_1^* \otimes \mathbb{1}_{\mathcal{A}})(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2^*)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} \\
&= \mathcal{V}^*(\mathcal{V}_1^* \otimes \mathbb{1}_{\mathcal{A}})(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_2^*\mathcal{V}_2)(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} = \mathcal{V}^*(\mathcal{V}_1^* \otimes \mathbb{1}_{\mathcal{A}})(\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} \\
&= \mathcal{V}^*(\mathcal{V}_1^*\mathcal{V}_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V} = \mathcal{V}^*\mathcal{V} = \mathbb{1}_{\mathcal{A}}.
\end{aligned}$$

Applying Lemma 1.3 we obtain that $\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}$ is a transition expectation. □

$\mathcal{E}_{\mathcal{V}_1,\mathcal{V}_2}$ is a model for an interaction of the quantum system with the measurement apparatus and additional independent evolutions of the system and the measurement equipment. We will use it in section 3.1.

For the description of repeated measurements we still need some more preparations.

Let for $j = 1, 2$ $\mathcal{U}_j$ be C^*-algebras and $\mathcal{F}_j : \mathcal{U}_j \otimes \mathcal{A} \longrightarrow \mathcal{A}$ transition expectations. Then we define a map $\mathcal{F}_1 \star \mathcal{F}_2 : \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{A} \longrightarrow \mathcal{A}$ by

$$(\mathcal{F}_1 \star \mathcal{F}_2)(U_1 \otimes U_2 \otimes A) := \mathcal{F}_1(U_1 \otimes \mathcal{F}_2(U_2 \otimes A)) \qquad (U_j \in \mathcal{U}_j, A \in \mathcal{A}).$$

Remark 1.5. *From the definition of $\mathcal{F}_1 \star \mathcal{F}_2$ we see that it is again a transition expectation, because the relevant properties of $\mathcal{F}_1$ and $\mathcal{F}_2$ are preserved.*

Now we will consider sequences of measurements.
Let $n \in \mathbb{N}$ and $\mathcal{B}_n$ be a C^*-algebra representing the observables of the measurement apparatus at the n-th measurement. Furthermore, let $\mathcal{E}_n : \mathcal{B}_n \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be a transition expectation. By $\mathcal{B}^{[k,n]}$, $k \leq n$, we denote the C^*-tensor product $\mathcal{B}_k \otimes \ldots \otimes \mathcal{B}_n$. Then we define $\mathcal{E}^{[k,n]} : \mathcal{B}^{[k,n]} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\mathcal{E}^{[k,n]}(B_k \otimes \ldots \otimes B_n \otimes A) := \mathcal{E}_k(B_k \otimes \mathcal{E}_{k+1}(B_{k+1} \otimes \ldots \otimes \mathcal{E}_n(B_n \otimes A) \ldots), \quad (1.1.4)$$

$$\mathcal{E}^{[k,k]} := \mathcal{E}_k, \qquad \mathcal{E}^{k]} := \mathcal{E}^{[1,k]}. \quad (1.1.5)$$

From this we get immediately

$$\mathcal{E}^{n+m]} = \mathcal{E}^{n]} \star \mathcal{E}^{[n+1,n+m]} = \mathcal{E}^{m]} \star \mathcal{E}^{[m+1,m+n]}.$$

For $\mathcal{B}_n = \mathcal{B}$, $\mathcal{E}_n = \mathcal{E}$ for all n we have $\mathcal{E}^{n]} = \mathcal{E}^{\star n}$.

1.2 The Quasilocal Algebra and Quantum Markov Chains

Now let $\mathcal{H}$ be a separable Hilbert space and for all $n \in \mathbb{N}$ $\mathcal{A}$ and $\mathcal{B}_n$ von Neumann subalgebras of $\mathfrak{L}(\mathcal{H})$. According to [8], chapter 2.7.2, there is a natural choice for a C^*-tensor product.
We will use transition expectations to construct quantum Markov chains, [3, 4].
We will see that in Definition 1.2, the second condition ensures that states are

mapped into states. The first one is necessary for mapping *normal* states into *normal* ones.

By $\Lambda_Q^{[k,n]} : \mathcal{A} \longrightarrow \mathcal{A}$

$$\begin{aligned}\Lambda_Q^{[k,n]}(A) &:= \mathcal{E}^{[k,n]}(\mathbb{1}_{\mathcal{B}^{[k,n]}} \otimes A) \\ &= \mathcal{E}^{[k,n]}(\mathbb{1}_{\mathcal{B}_k} \otimes \ldots \otimes \mathbb{1}_{\mathcal{B}_n} \otimes A) \\ &= \mathcal{E}_k(\mathbb{1}_{\mathcal{B}_k} \otimes \mathcal{E}_{k+1}(\mathbb{1}_{\mathcal{B}_{k+1}} \otimes \ldots \otimes \mathcal{E}_n(\mathbb{1}_{\mathcal{B}_n} \otimes A) \ldots)\end{aligned}$$

with $A \in \mathcal{A}$ we can describe the evolution of the quantum system from time k to n.
By $\Lambda_M^{[k,n]} : \mathcal{B}^{[k,n]} \longrightarrow \mathcal{A}$

$$\begin{aligned}\Lambda_M^{[k,n]}(B_1 \otimes \ldots \otimes B_n) &:= \mathcal{E}^{[k,n]}(B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}}) \\ &= \mathcal{E}^{[k,n]}(B_k \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}}) \\ &= \mathcal{E}_k(B_k \otimes \mathcal{E}_{k+1}(B_{k+1} \otimes \ldots \otimes \mathcal{E}_n(B_n \otimes \mathbb{1}_{\mathcal{A}}) \ldots)\end{aligned}$$

with $B_k \otimes \ldots \otimes B_n \in \mathcal{B}^{[k,n]}$ we can describe the evolution of the measurement apparatus from time k to n.
If an initial state $\tau \in \mathcal{S}(\mathcal{A})$ is given, the state $\omega_1 \in \mathcal{S}(\mathcal{B}_1)$,

$$\omega_1(B) := \tau \circ \Lambda_{\mathcal{B}_1,\mathcal{E}_1}(B) = \tau(\mathcal{E}_1(B \otimes \mathbb{1}_{\mathcal{A}})) \qquad (B \in \mathcal{B}_1)$$

describes the state of the measurement apparatus after the first measurement. For all $1 \leq k \leq n$ there is a state $\omega_{[k,n]}$ on $\mathcal{B}^{[k,n]}$ with

$$\omega_{[k,n]}(B_{[k,n]}) = \tau(\mathcal{E}^{[k,n]}(B_{[k,n]} \otimes \mathbb{1}_{\mathcal{A}})) \qquad (B_{[k,n]} \in \mathcal{B}^{[k,n]}),$$

which describes the measurement process between the k-th and the n-th measurement.

Now let $\mathcal{B}_n = \mathcal{B}$ for all $n \in \mathbb{N}$ and $\mathcal{B}^m$ for $m \in \mathbb{N}$ the m-times tensor product of $\mathcal{B}$. Let $\mathcal{C} = \bigotimes_{\mathbb{N}} \mathcal{B}$ the C^*-tensor product of a countable set of copies of $\mathcal{B}$. $\mathcal{C}$ has the following properties ([8], sec. 2.7.2):

(i) For each $n \in \mathbb{N}$ there exists an embedding $j_n : \mathcal{B} \longrightarrow \mathcal{C}$ (being the canonical identification of $\mathcal{B}$ with the n-th factor in $\mathcal{C}$) such that for all $m \in \mathbb{N}$ $j_{[1,m]} = j_1 \otimes \ldots \otimes j_m : \mathcal{B}^m \longrightarrow \mathcal{C}$ is a $*$ - isomorphism from $\mathcal{B}^m$ onto the range $j_{[1,m]}(\mathcal{B}^m)$ satisfying for all $B_1, \ldots, B_m \in \mathcal{B}$

$$\|j_{[1,m]}(B_1 \otimes \ldots \otimes B_m)\| = \|B_1\| \cdot \ldots \cdot \|B_m\|.$$

(ii) $\mathcal{C}$ is the norm closure of the algebra generated by all $j_n(B)$, $n \in \mathbb{N}$, $B \in \mathcal{B}$.

If we denote by J the family of finite subsets of $\mathbb{N}$ and for $I \in J$ by $\mathcal{C}_I$ the algebra generated by $\{j_n(B) : B \in \mathcal{B}, n \in I\}$ then for all $I_1, I_2 \in J$ with $I_1 \subseteq I_2$ we get $\mathcal{C}_{I_1} \subseteq \mathcal{C}_{I_2}$ and $\mathcal{C}_{I_1}$ and $\mathcal{C}_{I_2}$ commute whenever I_1 and I_2 are disjoint. Because of

$$\mathcal{C} = \overline{\bigcup_{I \in J} \mathcal{C}_I}$$

(see (ii)), the pair $[\mathcal{C}, (\mathcal{C}_I)_{I\in J}]$ represents a QUASILOCAL ALGEBRA in the sense of [8].
For each $I \in J$ the algebra $\mathcal{C}_I$ is isomorphic to $\mathcal{B}^{|I|}$, where $|I|$ is the cardinality of I. This means in particular that $\mathcal{C}_{n]}$ is isomorphic to $\mathcal{B}^n$.
The algebras $\mathcal{C}_I$ are called algebras of local oberservables or simply LOCAL ALGEBRAS.

Remark 1.6. *The embeddings of $\mathcal{B}^{[k,n]}$ in $\mathcal{C}$ represent the local algebras. The local states $\omega_{[k,n]}$ on $\mathcal{B}^{[k,n]}$ are compatible in the sense that for all $1 \le k \le m \le n$*

$$\omega_{[k,n]}(B_{[k,m]} \otimes \mathbb{1}_{\mathcal{B}^{[m+1,n]}}) = \omega_{[k,m]}(B_{[k,m]}) \qquad (B_{[k,m]} \in \mathcal{B}^{[k,m]}).$$

So, to each initial state $\tau \in \mathcal{S}(\mathcal{A})$ there exists a unique state $\omega \in \mathcal{S}(\mathcal{C})$ such that for all $n \in \mathbb{N}$

$$\omega_{n]}(B_{n]}) = \omega(B_{n]} \otimes \mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{B}} \otimes \ldots) \qquad (B_{n]} \in \mathcal{B}^{n]}). \tag{1.2.1}$$

ω is called quantum Markov chain:

Definition 1.7. *Let τ be a state on $\mathcal{A}$ and $(\mathcal{E}_n)_{n\in\mathbb{N}}$ a sequence of transition expectations from $\mathcal{B} \otimes \mathcal{A}$ to $\mathcal{A}$. The state ω on $\mathcal{C} = \bigotimes_{\mathbb{N}} \mathcal{B}$ defined uniquely by* (1.2.1) *with*

$$\omega_{n]} = \tau(\mathcal{E}^{n]}(. \otimes \mathbb{1}_{\mathcal{A}})) \tag{1.2.2}$$

and $\mathcal{E}^{n]}$ defined in (1.1.5) *is called the QUANTUM MARKOV CHAIN associated to the pair $(\tau, (\mathcal{E}_n)_{n=1}^{\infty})$. If for each n there holds $\mathcal{E}_n = \mathcal{E}$, then we speak of a HOMOGENEOUS quantum Markov chain. The state τ is called the INITIAL STATE, the $\mathcal{E}_n$ are called the transition expectations of the quantum Markov chain.*

Normal states represent states of finite boson systems.

Definition 1.8. *A state $\omega \in \mathcal{S}(\mathcal{B})$ is called NORMAL STATE if there exists a DENSITY MATRIX K, i.e. a positive trace class operator on $\mathcal{H}$ with $TrK = 1$ such that*

$$\omega(A) = Tr(KA)$$

for all $A \in \mathcal{B}$.

ω is normal if and only if it is σ-weakly continuous ([8], Theorem 2.4.21).

Locally normal states describe states of infinite boson systems that are locally finite. The restriction of a locally normal state to a local algebra $\mathcal{C}_I$ can be identified with a normal state on the Fock space over the bounded region corresponding to the index set I.

Definition 1.9. *A state $\omega \in \mathcal{S}(\mathcal{C})$ is called LOCALLY NORMAL STATE if for each $n \in \mathbb{N}$ the restriction of ω to $\mathcal{C}_{n]}$ is a normal state, i.e. there exists a normal state $\omega_{n]} \in \mathcal{S}(\mathcal{B}^n)$ such that*

$$\omega\left(\prod_{i=1}^{n} j_i(B_i)\right) = \omega_{n]}\left(\bigotimes_{i=1}^{n} B_i\right) \qquad (B_i \in \mathcal{B},\ i \in n]).$$

Remark 1.10. *If τ is a normal state on $\mathcal{A}$, then the quantum Markov chain associated with τ and $(\mathcal{E}_n)_{n=1}^{\infty}$ is a locally normal state on $\mathcal{C}$.*

In the following we will look for normal and locally normal states that are invariant under a certain mapping.

Definition 1.11. *Let $\mathcal{A}$ be a C^*-algebra, $\omega \in \mathcal{S}(\mathcal{A})$ and $\Lambda : \mathcal{A} \longrightarrow \mathcal{A}$ a linear mapping.*
Then ω is called INVARIANT STATE under the mapping Λ if

$$\omega \circ \Lambda = \omega. \tag{1.2.3}$$

Chapter 2

Preparations

2.1 The Boson Fock Space

Let G be an arbitrary complete separable metric space and $\mathfrak{G}$ the associated σ-algebra of BOREL sets from G. The ring of all bounded BOREL sets from $\mathfrak{G}$ is denoted by $\mathfrak{B}$. Moreover, let ν be a locally finite measure on $[G, \mathfrak{G}]$, i.e. $\nu(B) < \infty$ for all $B \in \mathfrak{B}$. By $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we denote the set of all natural numbers including zero. Let M be the set of all locally finite counting measures on $[G, \mathfrak{G}]$, i.e.

$$M := \{\varphi : \varphi \text{ is a measure on } [G, \mathfrak{G}], \varphi(B) \in \mathbb{N}_0 \text{ for all } B \in \mathfrak{B}\}.$$

Each $\varphi \in M$ is of the form $\varphi = \sum\limits_{j \in J} \delta_{x_j}$ with an at most countable index set J, $x_j \in G$ for all $j \in J$ and the sequence $(x_j)_{j \in J}$ having no accumulation points. δ_x denotes the Dirac measure in x. Each element of M describes a locally finite point configuration in G. Multiple points are allowed, for example $\varphi = 2\delta_{x_1} + 3\delta_{x_2}$ with x_1, $x_2 \in G$ is a valid point configuration. For two counting measures $\hat{\varphi}$, $\varphi \in M$ we write $\hat{\varphi} \subseteq \varphi$, if for all $B \in \mathfrak{B}$ there holds $\hat{\varphi}(B) \leq \varphi(B)$. This means that $\hat{\varphi}$ is a subconfiguration of φ.
By *supp* $\varphi := \{x \in G : \varphi(\{x\}) > 0\}$ we denote the support of $\varphi \in M$.
Let for $\varphi \in M$ $|\varphi| := \varphi(G)$ be the number of points in the configuration φ (including multiplicities).
M^f denotes the set of all finite point configurations:

$$M^f := \{\varphi \in M : |\varphi| < \infty\}.$$

The set $\{\varphi \in M : |\varphi| = n\}$ of all n-point counting measures from M for $n \in \mathbb{N}$ is denoted by M_n, hence, $M^f = \bigcup\limits_{n=0}^{\infty} M_n$. Here, $M_0 = \{\mathfrak{o}\}$ is the set containing only the zero measure $\mathfrak{o}$, i.e. the empty configuration in M ($\mathfrak{o}(G) = 0$).
By $M^m := \{\underline{\varphi} = (\varphi_1, \ldots, \varphi_m), \varphi_i \in M \forall i \in m]\}$ we denote the set of all m-dimensional vectors with components for M.
The σ-algebra generated by all sets of the type $\{\varphi \in M : \varphi(B) = n\}$, $B \in \mathfrak{B}$, $n \in \mathbb{N}_0$, is denoted by $\mathfrak{M}$. It is the smallest σ-algebra making the map $\varphi \mapsto \varphi(B)$ measurable for all $B \in \mathfrak{B}$.

Definition 2.1. *A probability measure on $[M, \mathfrak{M}]$ is called a POINT PROCESS.*

Usually such a measure is interpreted as distribution of a random point configuration in G. An important example for a point process is the POISSON point process. For details see [9], [25].

Definition 2.2. *Let P be a point process on $[M, \mathfrak{M}]$ and λ a locally finite measure on $[G, \mathfrak{G}]$. P is called POISSON POINT PROCESS with intensity measure λ if for all $m \in \mathbb{N}$, pairwise disjoint $B_1, \dots, B_m \in \mathfrak{B}$ and $k_1, \dots, k_m \in \mathbb{N}$*

$$P(\{\varphi : \varphi(B_1) = k_1, \dots, \varphi(B_m) = k_m\}) = \prod_{j=1}^{m} exp\Big(-\lambda(B_j)\Big) \cdot \frac{\lambda(B_j)^{k_j}}{k_j!}. \qquad (2.1.1)$$

Now we define a σ-finite measure F on $[M, \mathfrak{M}]$.

Definition 2.3. *For $Y \in \mathfrak{M}$ let*

$$F(Y) := \chi_Y(\mathfrak{o}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{G^n} \nu^n(d[x_1, \dots x_n]) \chi_Y\Big(\sum_{j=1}^{n} \delta_{x_j}\Big). \qquad (2.1.2)$$

Here χ_Y denotes the indicator function of a set $Y \in \mathfrak{M}$.

Definition 2.4. *The space $\mathcal{M} := \mathcal{L}^2(M, \mathfrak{M}, F)$ is called the (SYMMETRIC) FOCK SPACE over G according to the reference measure ν.*

$\mathcal{M}$ is again a separable Hilbert space.
For all $n \in \mathbb{N}$ let $\mathcal{M}^n := \mathcal{M}^{\bigotimes n}$ be the n-fold tensor product of the Hilbert space $\mathcal{M}$. Obviously, $\mathcal{M}^n$ can be identified with $\mathcal{L}^2(M^n, \mathfrak{M}^n, F^n)$.
By $\langle .,. \rangle_{\mathcal{M}^n}$ we will denote the scalar product in $\mathcal{L}^2(M^n, \mathfrak{M}^n, F^n)$.

Remark 2.5. *Usually, the symmetric Fock space over $\mathcal{H}$ is defined as follows.
Let $\mathcal{H}$ be a separable Hilbert space. Then the symmetric Fock space over $\mathcal{H}$ is defined by*

$$\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathcal{H}_{sym}^{\otimes n}$$

*where $\mathcal{H}_{sym}^{\otimes 0} := \mathbb{C}$ and for $n \in \mathbb{N}$ $\mathcal{H}_{sym}^{\otimes n}$ is the n-fold symmetric tensor product of $\mathcal{H}$, i.e. the Hilbert subspace of $\mathcal{H}^{\otimes n}$ generated by vectors $u^{\otimes n}$, $u \in \mathcal{H}$ arbitrary.
Consequently,*

$$\Gamma(\mathcal{H}) = \Big\{(f_n)_{n \in \mathbb{N}_0} : f_n \in \mathcal{H}_{sym}^{\otimes n}, \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \|f_n\|^2 < \infty \Big\}.$$

The space $\mathcal{M}$ given in Definition 2.4 is isomorphic to $\Gamma(\mathcal{L}^2(G, \nu))$ under the isomorphism I determined by

$$(Iu^{\otimes n})(\varphi) := \delta_{n, |\varphi|} \prod_{x \in supp \varphi} u(x)^{\varphi(\{x\})}$$

*where $\delta_{i,j}$ denotes the Kronecker delta symbol.
For further details and proof see [28].*

2.2 Generalized Binomial Coefficients and $\sum\!\!\!\!\!\!\int$ -Lemma

In many publications like [19] and [24] there are considered only diffuse measures ν on $[G, \mathfrak{G}]$, i.e. $\nu(x) = 0$ for all singletons $x \in G$. In this work, like in [32], also reference measures ν with atoms are allowed. For the point configurations this means that there may exist multiple points, i.e. $\varphi = \sum_{j \in J} k_j \delta_{x_j}$ with $k_j > 1$. Therefore, some formulae must be supplemented by additional factors.

Definition 2.6. *Let $\varphi, \hat{\varphi} \in M$ with $\hat{\varphi} \subseteq \varphi$. We define the number $\binom{\varphi}{\hat{\varphi}} \in \mathbb{N}$ by*

$$\binom{\varphi}{\hat{\varphi}} := \begin{cases} \prod\limits_{x \in supp \hat{\varphi}} \binom{\varphi(\{x\})}{\hat{\varphi}(\{x\})} & , \ \hat{\varphi} \neq \mathfrak{o}, \\ 1 & , \ \hat{\varphi} = \mathfrak{o}. \end{cases} \tag{2.2.1}$$

with $\binom{k}{l}$ being the usual binomial coefficient.

Remark 2.7. *We could as well use the definition*

$$\binom{\varphi}{\widehat{\varphi}} := \prod_{x \in supp \ \varphi} \binom{\varphi(\{x\})}{\hat{\varphi}(\{x\})} \ , \quad \varphi \neq \mathfrak{o}$$

where we form the product over all $x \in$ supp φ. (For $x \notin$ supp $\widehat{\varphi}$ we get factors $\binom{\varphi(\{x\})}{0} = 1$.) Then we have to set additionally $\binom{\mathfrak{o}}{\mathfrak{o}} := 1$. For countable phase space G we could even use

$$\binom{\varphi}{\widehat{\varphi}} := \prod_{x \in G} \binom{\varphi(\{x\})}{\widehat{\varphi}(\{x\})} \ , \quad \varphi \neq \mathfrak{o},$$

defining again $\binom{\mathfrak{o}}{\mathfrak{o}} := 1$.

Now we give some properties of these generalized binomial coefficients.
For $B \in \mathfrak{G}$ we denote by $B^c := G \setminus B$ the complement of B.

Lemma 2.8. *Let $n \in \mathbb{N}$, $\varphi, \varphi_1, ..., \varphi_n \in M$. Furthermore, let for Borel sets $B \in \mathfrak{B}$ $\varphi_{|_B}(.) := \varphi(B \cap (.))$ be the restriction of φ to B. Then there holds*

(a)

$$\binom{\varphi_1 + \varphi_2}{\varphi_1} = \binom{\varphi_1 + \varphi_2}{\varphi_2}.$$

(b) For $B \in \mathfrak{B}$ and $\varphi_1 \subseteq \varphi_{|_B}$, $\varphi_2 \subseteq \varphi_{|_{B^c}}$ there holds

$$\binom{\varphi}{\varphi_1 + \varphi_2} = \binom{\varphi_{|_B} + \varphi_{|_{B^c}}}{\varphi_1 + \varphi_2} = \binom{\varphi_{|_B}}{\varphi_1} \cdot \binom{\varphi_{|_{B^c}}}{\varphi_2}.$$

(c)
$$\prod_{k=1}^{n-1}\binom{\varphi_1+\ldots+\varphi_{k+1}}{\varphi_1+\ldots+\varphi_k}=\prod_{x\in G}\frac{[(\varphi_1+\ldots+\varphi_n)(\{x\})]!}{[\varphi_1(\{x\})]!\cdot\ldots\cdot[\varphi_n(\{x\})]!}.$$

PROOF. (a) For $\varphi_1=\mathfrak{o}$ or $\varphi_2=\mathfrak{o}$ there is nothing to prove.
Let $\varphi_1\neq\mathfrak{o}$ and $\varphi_2\neq\mathfrak{o}$. For $i,\,j\in\mathbb{N}$ we have $\binom{i+j}{i}=\binom{i+j}{j}$. Using this, the definition of the generalized binomial coefficients and Remark 2.7, we get

$$\begin{aligned}\binom{\varphi_1+\varphi_2}{\varphi_1}&=\prod_{x\in supp(\varphi_1+\varphi_2)}\binom{(\varphi_1+\varphi_2)(\{x\})}{\varphi_1(\{x\})}=\prod_{x\in supp(\varphi_1+\varphi_2)}\binom{(\varphi_1(\{x\})+\varphi_2(\{x\})}{\varphi_1(\{x\})}\\&=\prod_{x\in supp(\varphi_1+\varphi_2)}\binom{(\varphi_1(\{x\})+\varphi_2(\{x\})}{\varphi_2(\{x\})}=\prod_{x\in supp(\varphi_1+\varphi_2)}\binom{(\varphi_1+\varphi_2)(\{x\})}{\varphi_2(\{x\})}\\&=\binom{\varphi_1+\varphi_2}{\varphi_2}.\end{aligned}$$

(b) Because of $B\cup B^c=G$ and $B\cap B^c=\emptyset$ there holds $\varphi=\varphi_{|B}+\varphi_{|B^c}$ for all $\varphi\in M$. So for $\varphi_1\subseteq\varphi_{|B}$ and $\varphi_2\subseteq\varphi_{|B^c}$ we have $supp(\varphi_1+\varphi_2)=supp(\varphi_1)\cup supp(\varphi_2)$ and $supp(\varphi_1)\cap supp(\varphi_2)=\emptyset$, i.e., $x\in supp(\varphi_1)\Rightarrow x\notin supp(\varphi_2)$ and $x\in supp(\varphi_2)\Rightarrow x\notin supp(\varphi_1)$. Therefore we have

$$\begin{aligned}\binom{\varphi}{\varphi_1+\varphi_2}&=\binom{\varphi_{|B}+\varphi_{|B^c}}{\varphi_1+\varphi_2}=\prod_{x\in supp(\varphi_1+\varphi_2)}\binom{\varphi_{|B}(\{x\})+\varphi_{|B^c}(\{x\})}{\varphi_1(\{x\})+\varphi_2(\{x\})}\\&=\prod_{x\in supp\,\varphi_1}\binom{\varphi_{|B}(\{x\})+\varphi_{|B^c}(\{x\})}{\varphi_1(\{x\})+\varphi_2(\{x\})}\cdot\prod_{x\in supp\,\varphi_2}\binom{\varphi_{|B}(\{x\})+\varphi_{|B^c}(\{x\})}{\varphi_1(\{x\})+\varphi_2(\{x\})}\\&=\prod_{x\in supp\,\varphi_1}\binom{\varphi_{|B}(\{x\})}{\varphi_1(\{x\})}\cdot\prod_{x\in supp\,\varphi_2}\binom{\varphi_{|B^c}(\{x\})}{\varphi_2(\{x\})}=\binom{\varphi_{|B}}{\varphi_1}\cdot\binom{\varphi_{|B^c}}{\varphi_2}.\end{aligned}$$

(c) We know that for $k\in\mathbb{N}$, $n_1,\ldots,n_k\in\mathbb{N}_0$

$$\binom{n_1+\cdots+n_k}{n_1+\ldots+n_{k-1}}\cdot\binom{n_1+\ldots+n_{k-1}}{n_1+\ldots+n_{k-2}}\cdot\ldots\cdot\binom{n_1+n_2}{n_1}=\frac{(n_1+\ldots+n_k)!}{n_1!\cdot\ldots\cdot n_k!}.$$

For $\varphi_1,\ldots,\varphi_n\in M$ we have

$$\begin{aligned}&\prod_{k=1}^{n-1}\binom{\varphi_1+\ldots+\varphi_{k+1}}{\varphi_1+\ldots+\varphi_k}\\&=\prod_{x\in supp\,\varphi}\binom{(\varphi_1+\ldots+\varphi_n)(\{x\})}{(\varphi_1+\ldots+\varphi_{n-1})(\{x\})}\binom{(\varphi_1+\ldots+\varphi_{n-1})(\{x\})}{(\varphi_1+\ldots+\varphi_{n-2})(\{x\})}\cdots\binom{(\varphi_1+\varphi_2)(\{x\})}{\varphi_1(\{x\})}\\&=\prod_{x\in supp\,\varphi}\Bigg(\frac{[(\varphi_1+...+\varphi_n)(\{x\})]!}{[\varphi_n(\{x\})]!\cdot[(\varphi_1+...+\varphi_{n-1})(\{x\})]!}\\&\qquad\cdot\frac{[(\varphi_1+...+\varphi_{n-1})(\{x\})]!}{[\varphi_{n-1}(\{x\})]!\cdot[(\varphi_1+...+\varphi_{n-2})(\{x\})]!}\cdot\;\ldots\cdot\frac{[(\varphi_1+\varphi_2)(\{x\})]!}{[\varphi_2(\{x\})]!\cdot[\varphi_1(\{x\})]!}\Bigg)\end{aligned}$$

$$= \prod_{x \in supp\,\varphi} \frac{[(\varphi_1 + \ldots + \varphi_n)(\{x\})]!}{[\varphi_1(\{x\})]! \cdot \ldots \cdot [\varphi_n(\{x\})]!}.$$

$\square$

For comfortable working with functions of point configurations from M we need some more preparations.

Lemma 2.9. *For functions $f : M \times M \longrightarrow \mathbb{C}$, $m, n \in \mathbb{N}$, $m \leq n$ and $\varphi \in M_n$, $\varphi = \sum_{j=1}^{n} \delta_{x_j}$, $x_j \in G$, there holds*

$$\sum_{\substack{\widehat{\varphi} \subseteq \varphi \\ |\widehat{\varphi}| = m}} \binom{\varphi}{\widehat{\varphi}} f(\widehat{\varphi}, \varphi - \widehat{\varphi}) \quad = \sum_{\{i_1 \ldots i_m\} \subseteq \{1, \ldots n\}} f(\sum_{j=1}^{m} \delta_{x_{i_j}}, \varphi - \sum_{j=1}^{m} \delta_{x_{i_j}}). \tag{2.2.2}$$

PROOF. By definition, $\varphi \in M_n$ is of the form $\varphi = \sum_{j=1}^{n} \delta_{x_j}$. Since in this representation not all x_j are necessarily distinct, we rewrite it:

$$\varphi = \sum_{j=1}^{l} k_j \delta_{y_j} \text{ with } y_1, \ldots y_l \text{ distinct },\ l \leq n\ \ , k_j \geq 1\ \forall j = 1, \ldots, l\ \ ,\ \sum_{j=1}^{l} k_j = n.$$

Let $\widehat{\varphi} \subseteq \varphi$, $|\widehat{\varphi}| = m$. $\widehat{\varphi}$ is therefore of the form

$$\widehat{\varphi} = \sum_{j=1}^{l} m_j \delta_{y_j} \text{ with } 0 \leq m_j \leq k_j\ \forall j = 1, \ldots, l\ \ ,\ \sum_{j=1}^{l} m_j = m.$$

Without loss of generality we may assume

$$\begin{aligned} x_1 = x_2 = \ldots = x_{k_1} &= y_1, \\ x_{k_1+1} = x_{k_1+2} = \ldots = x_{k_1+k_2} &= y_2, \\ &\vdots \\ x_{\sum_{j=1}^{l-1} k_j + 1} = x_{(\sum_{j=1}^{l-1} k_j) + 2} = \ldots = x_{\sum_{j=1}^{l} k_j} &= y_l. \end{aligned}$$

Using these representations of φ and $\widehat{\varphi}$ we get for $f : M \times M \longrightarrow \mathbb{C}$

$$\sum_{\substack{\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \\ \sum_{j=1}^{m} \delta_{x_{i_j}} = \widehat{\varphi}}} f(\hat{\varphi}, \varphi - \hat{\varphi}) = \sum_{\substack{\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \\ \sum_{j=1}^{m} \delta_{x_{i_j}} = \widehat{\varphi}}} f(\sum_{j=1}^{m} \delta_{x_{i_j}}, \varphi - \sum_{j=1}^{m} \delta_{x_{i_j}})$$

$$= \sum_{\substack{\{i_1, \ldots, i_{m_1}\} \subseteq \{1, \ldots, k_1\} \\ \{i_{m_1+1}, \ldots, i_{m_1+m_2}\} \subseteq \{k_1+1, \ldots, k_1+k_2\} \\ \vdots \\ \{i_{(\sum_{j=1}^{l-1} m_j)+1}, \ldots, m\} \subseteq \{(\sum_{j-1}^{l-1} k_j)+1, \ldots, n\}}} f(\sum_{j=1}^{m} \delta_{x_{i_j}}, \varphi - \sum_{j=1}^{m} \delta_{x_{i_j}})$$

$$= \binom{k_1}{m_1}\binom{k_2}{m_2}\cdots\binom{k_l}{m_l} f(\widehat{\varphi}, \varphi - \widehat{\varphi}) = \binom{\varphi}{\widehat{\varphi}} f(\widehat{\varphi}, \varphi - \widehat{\varphi}).$$

This implies

$$\sum_{\substack{\widehat{\varphi}\subseteq\varphi \\ |\widehat{\varphi}|=m}} \binom{\varphi}{\widehat{\varphi}} f(\widehat{\varphi}, \varphi - \widehat{\varphi}) = \sum_{\substack{\hat{\varphi}\subseteq\varphi \\ |\hat{\varphi}|=m}} \sum_{\substack{\{i_1,\dots,i_m\}\subseteq\{1,\dots,n\} \\ \sum_{j=1}^{m} \delta_{x_{i_j}} = \hat{\varphi}}} f(\hat{\varphi}, \varphi - \hat{\varphi})$$

$$= \sum_{\{i_1,\dots,i_m\}\subseteq\{1,\dots,n\}} f\Big(\sum_{j=1}^{m} \delta_{x_{i_j}}, \varphi - \sum_{j=1}^{m} \delta_{x_{i_j}}\Big).$$

□

Corollary 2.10. *For all functions $h : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$ and all $\varphi \in M^f$ there holds*

$$\sum_{\widehat{\varphi}\subseteq\varphi} \binom{\varphi}{\widehat{\varphi}} h(|\widehat{\varphi}|, |\varphi - \widehat{\varphi}|) = \sum_{k=0}^{|\varphi|} \binom{|\varphi|}{k} h(k, |\varphi| - k). \tag{2.2.3}$$

Particularly, for all $a, b \in \mathbb{C}$ we have

$$\sum_{\widehat{\varphi}\subseteq\varphi} \binom{\varphi}{\widehat{\varphi}} a^{|\widehat{\varphi}|} b^{|\varphi-\widehat{\varphi}|} = (a+b)^{|\varphi|}. \tag{2.2.4}$$

PROOF. From Lemma 2.9 we get also for $h : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$ and $\varphi \in M^n$

$$\sum_{\substack{\widehat{\varphi}\subseteq\varphi \\ |\widehat{\varphi}|=m}} \binom{\varphi}{\widehat{\varphi}} h(|\widehat{\varphi}|, |\varphi - \widehat{\varphi}|) = \sum_{\{i_1\dots i_m\}\subseteq\{1,\dots n\}} h(m, n-m) = \binom{n}{m} h(m, n-m). \tag{2.2.5}$$

By summing up over all $m \in \{0, \dots, n\}$ we get the first statement. The second statement follows immediately from the first one by applying the Binomial Theorem.

□

The following proposition will be the basis for many proofs, replacing integration over M^n (with respect to F^n) by integration over M (with respect to F).
The parts (b) for $n = 2$ and (d) for arbitrary $n \in \mathbb{N}$, $n \geq 2$ are often called $\sum\!\!\!\!\!\!\int$ - Lemma. The proof for $n = 2$ was for instance given in [30], for the case of reference measure ν without atoms see also [17], [24], [35], [36].
We will give the complete proof for the general case of atomic reference measure ν and arbitrary $n \in \mathbb{N}$, $n \geq 2$.

Proposition 2.11.

(a) Let $n, m \in \mathbb{N}_0$, $h : M \times M \longrightarrow \mathbb{C}$ be a function integrable with respect to F^2 (or ≥ 0). Then

$$\int_{M_n} F(d\varphi_2) \int_{M_m} F(d\varphi_1) h(\varphi_1, \varphi_2) = \int_{M_{m+n}} F(d\varphi) \sum_{\substack{\hat{\varphi}\subseteq\varphi \\ |\varphi|=m}} \binom{\varphi}{\hat{\varphi}} h(\hat{\varphi}, \varphi - \hat{\varphi}).$$

(b) Let $h : M \times M \longrightarrow \mathbb{C}$ be a function integrable with respect to F^2 (or ≥ 0). Then

$$\int F(d\varphi_1)\int F(d\varphi_2) h(\varphi_1,\varphi_2) = \int F(d\varphi)\sum_{\hat{\varphi}\subseteq\varphi}\binom{\varphi}{\hat{\varphi}} h(\hat{\varphi},\varphi-\hat{\varphi}).$$

(c) Let $n \geq 2$, $m_i \in \mathbb{N}$ for $i \in n]$ and $f : M^n \longrightarrow \mathbb{C}$ be a function integrable with respect to F^n (or ≥ 0). Then

$$\begin{aligned}
&\int\limits_{M_{m_n}} \cdots \int\limits_{M_{m_1}} F^n(d[\varphi_1,\ldots\varphi_n]) f(\varphi_1,\ldots,\varphi_n)\\
&= \int\limits_{M_{m_1+\ldots+m_n}} F(d\varphi_n) \sum_{\substack{\varphi_1\subseteq\ldots\subseteq\varphi_n\\ |\varphi_1|=m_1\\ \vdots \\ |\varphi_{n-1}|=m_1+\ldots m_{n-1}}} \binom{\varphi_n}{\varphi_{n-1}}\cdots\binom{\varphi_2}{\varphi_1} f(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1}).
\end{aligned}$$

(d) Let $n \geq 2$ and $f : M^n \longrightarrow \mathbb{C}$ be a function integrable with respect to F^n (or ≥ 0). Then

$$\begin{aligned}
&\int \cdots \int F^n(d[\varphi_1,\ldots\varphi_n]) f(\varphi_1,\ldots,\varphi_n)\\
&= \int F(d\varphi_n) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}}\binom{\varphi_{n-1}}{\varphi_{n-2}}\cdots\binom{\varphi_2}{\varphi_1} f(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\\
&= \int F(d\varphi) \sum_{\substack{\varphi_1,\ldots,\varphi_n:\\ \varphi_1+\ldots+\varphi_n=\varphi}} \prod_{x\in G} \frac{[(\varphi_1+\ldots+\varphi_n)(\{x\})]!}{[\varphi_1(\{x\})]!\cdot\ldots\cdot[\varphi_n(\{x\})]!}\cdot f(\varphi_1,\ldots,\varphi_n).
\end{aligned}$$

PROOF. (a) Let $\varphi \in M_{m+n}$. Then $\varphi = \sum\limits_{j=1}^{n+m}\delta_{x_j}$, with not all x_j being necessarily distinct. Using Lemma 2.9 we get

$$\begin{aligned}
&\int\limits_{M_{m+n}} F(d\varphi)\sum_{\substack{\hat{\varphi}\subseteq\varphi\\ |\hat{\varphi}|=m}}\binom{\varphi}{\hat{\varphi}} h(\hat{\varphi},\varphi-\hat{\varphi})\\
&= \frac{1}{(m+n)!}\int\limits_{G^{m+n}} \nu(dx_1)\ldots\nu(dx_{m+n}) \sum_{\substack{\{i_1,\ldots,i_m\}\\ \subseteq\{1,\ldots,n+m\}}} h(\sum_{j=1}^m \delta_{x_{i_j}}, \sum_{k=1}^{m+n}\delta_{x_k} - \sum_{j=1}^m\delta_{x_{i_j}})\\
&= \frac{1}{(m+n)!}\sum_{\substack{\{i_1,\ldots,i_m\}\\ \subseteq\{1,\ldots,n+m\}}} \int\limits_{G^{m+n}} \nu(dx_1)\ldots\nu(dx_{m+n}) h(\sum_{j=1}^m \delta_{x_{i_j}}, \sum_{k=1}^{m+n}\delta_{x_k} - \sum_{j=1}^m\delta_{x_{i_j}})\\
&= \frac{1}{(m+n)!}\binom{m+n}{m}\int\limits_{G^{m+n}} \nu(dx_1)\ldots\nu(dx_{m+n}) h(\sum_{j=1}^m \delta_{x_j}, \sum_{k=m+1}^{m+n}\delta_{x_k})
\end{aligned}$$

$$= \frac{1}{n!}\int\limits_{G^n} \nu(dx_{m+1})\dots\nu(dx_{m+n})\frac{1}{m!}\int\limits_{G^m}\nu(dx_1)\dots\nu(dx_m)h(\sum_{j=1}^{m}\delta_{x_j},\sum_{k=m+1}^{m+n}\delta_{x_k})$$
$$= \int\limits_{M_n} F(d\varphi_2)\int\limits_{M_m} F(d\varphi_1)h(\varphi_1,\varphi_2).$$

This proves (a).

(b) According to definition there holds $M^f = \bigcup\limits_{n=0}^{\infty} M_n$ and $F(M\setminus M^f) = 0$. From this and (a) we conclude

$$\int\limits_M F(d\varphi_2)\int\limits_M F(d\varphi_1)h(\varphi_1,\varphi_2) = \sum_{n,m=0}^{\infty}\int\limits_{M_n} F(d\varphi_2)\int\limits_{M_m} F(d\varphi_1)h(\varphi_1,\varphi_2)$$
$$= \sum_{m,n=0}^{\infty}\int\limits_{M_{m+n}} F(d\varphi)\sum_{\substack{\hat\varphi\subseteq\varphi\\|\hat\varphi|=m}}\binom{\varphi}{\hat\varphi}h(\hat\varphi,\varphi-\hat\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\int\limits_{M_n} F(d\varphi)\sum_{\substack{\hat\varphi\subseteq\varphi\\|\hat\varphi|=m}}\binom{\varphi}{\hat\varphi}h(\hat\varphi,\varphi-\hat\varphi)$$
$$= \sum_{n=0}^{\infty}\int\limits_{M_n} F(d\varphi)\sum_{m=0}^{n}\sum_{\substack{\hat\varphi\subseteq\varphi\\|\hat\varphi|=m}}\binom{\varphi}{\hat\varphi}h(\hat\varphi,\varphi-\hat\varphi) = \sum_{n=0}^{\infty}\int\limits_{M_n} F(d\varphi)\sum_{\hat\varphi\subseteq\varphi}\binom{\varphi}{\hat\varphi}h(\hat\varphi,\varphi-\hat\varphi)$$
$$= \int\limits_M F(d\varphi)\sum_{\hat\varphi\subseteq\varphi}\binom{\varphi}{\hat\varphi}h(\hat\varphi,\varphi-\hat\varphi).$$

Hence, (b) holds true.

(c) For $n = 2$ we get the statement immediately from (a).
Now let for $n \geq 2$ and $f : M^{n-1} \longrightarrow \mathbb{C}$

$$\int\limits_{M_{m_{n-1}}} \dots \int\limits_{M_{m_1}} F^{n-1}(d[\varphi_1,\dots\varphi_{n-1}])f(\varphi_1,\dots,\varphi_{n-1}) \tag{2.2.6}$$
$$= \int\limits_{M_{m_1+\dots+m_{n-1}}} F(d\varphi_{n-1})\sum_{\substack{\varphi_1\subseteq\dots\subseteq\varphi_{n-1}\\|\varphi_1|=m_1\\\vdots\\|\varphi_{n-2}|=m_1+\dots+m_{n-2}}}\binom{\varphi_{n-1}}{\varphi_{n-2}}\dots\binom{\varphi_2}{\varphi_1}f(\varphi_1,\varphi_2-\varphi_1,\dots,\varphi_{n-1}-\varphi_{n-2}).$$

From this, we get using (a) and (2.2.6)

$$\int\limits_{M_{m_1+\dots+m_n}} F(d\varphi_n)\sum_{\substack{\varphi_1\subseteq\dots\subseteq\varphi_n\\|\varphi_1|=m_1\\\vdots\\|\varphi_{n-1}|=m_1+\dots+m_{n-1}}}\binom{\varphi_n}{\varphi_{n-1}}\dots\binom{\varphi_2}{\varphi_1}f(\varphi_1,\varphi_2-\varphi_1,\dots,\varphi_n-\varphi_{n-1})$$
$$= \int\limits_{M_{m_1+\dots+m_n}} F(d\varphi_n)\sum_{\substack{\varphi_{n-1}\subseteq\varphi_n\\|\varphi_{n-1}|=m_1+\dots+m_{n-1}}}\binom{\varphi_n}{\varphi_{n-1}}$$

$$\sum_{\substack{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}\\ |\varphi_1|=m_1\\ \vdots \\ |\varphi_{n-2}|=m_1+\ldots+m_{n-2}}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots, \varphi_n-\varphi_{n-1})$$

$$= \int\limits_{M_{m_n}} F(d\varphi_n) \int\limits_{M_{m_1+\ldots+m_{n-1}}} F(d\varphi_{n-1}) \sum_{\substack{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}\\ |\varphi_1|=m_1\\ \vdots \\ |\varphi_{n-2}|=m_1+\ldots+m_{n-2}}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots, \varphi_{n-1}-\varphi_{n-2}, \varphi_n)$$

$$= \int\limits_{M_{m_n}} F(d\varphi_n) \Bigg(\int\limits_{M_{m_{n-1}}} F(d\varphi_{n-1}) \ldots \int\limits_{M_{m_1}} F(d\varphi_1) f(\varphi_1, \ldots, \varphi_n) \Bigg)$$

$$= \int\limits_{M_{m_n}} \ldots \int\limits_{M_{m_1}} F^n(d[\varphi_1, \ldots \varphi_n]) f(\varphi_1, \ldots, \varphi_n).$$

hence, (c) is shown.
(d) We will prove this formula by induction. For $n = 2$ the statement follows immediately from (b).
Now let for $n \geq 2$ and $f : M^{n-1} \longrightarrow \mathbb{C}$

$$\int F(d\varphi_{n-1}) \ldots \int F(d(\varphi_1) f(\varphi_1, \ldots, \varphi_{n-1}) \tag{2.2.7}$$
$$= \int F(d\varphi_{n-1}) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots \varphi_{n-1}-\varphi_{n-2}).$$

Using (2.2.7) and (a) we obtain for $f : M^n \longrightarrow \mathbb{C}$

$$\int F(d\varphi_n) \ldots \int F(d\varphi_1) f(\varphi_1, \ldots, \varphi_n)$$
$$= \int F(d\varphi_n) \int F(d\varphi_{n-1}) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots \varphi_{n-1}-\varphi_{n-2}, \varphi_n)$$
$$= \sum_{m,n=0}^{\infty} \int\limits_{M_n} F(d\varphi_n) \int\limits_{M_m} F(d\varphi_{n-1}) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots, \varphi_{n-1}-\varphi_{n-2}, \varphi_n)$$
$$= \sum_{m,n=0}^{\infty} \int\limits_{M_{n+m}} F(d\varphi_n) \sum_{\substack{\varphi_{n-1}\subseteq\varphi_n\\ |\varphi_{n-1}|=m}} \binom{\varphi_n}{\varphi_{n-1}} \cdot$$
$$\cdot \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2-\varphi_1, \ldots, \varphi_n-\varphi_{n-1})$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_{M_n} F(d\varphi_n) \sum_{\substack{\varphi_{n-1} \subseteq \varphi_n \\ |\varphi_{n-1}|=m}} \binom{\varphi_n}{\varphi_{n-1}} \cdot$$

$$\cdot \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1})$$

$$= \sum_{n=0}^{\infty} \int_{M_n} F(d\varphi_n) \sum_{m=0}^{n} \sum_{\substack{\varphi_{n-1} \subseteq \varphi_n \\ |\varphi_{n-1}|=m}} \binom{\varphi_n}{\varphi_{n-1}} \cdot$$

$$\cdot \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1})$$

$$= \sum_{n=0}^{\infty} \int_{M_n} F(d\varphi_n) \sum_{\varphi_{n-1} \subseteq \varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \cdot$$

$$\cdot \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1})$$

$$= \int_M F(d\varphi_n) \sum_{\varphi_{n-1} \subseteq \varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1})$$

$$= \int_M F(d\varphi_n) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \ldots \binom{\varphi_2}{\varphi_1} f(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}).$$

Therefore, the first equation in (d) is shown. The second identity is a reformulation of the first one using part (c) of Lemma 2.8. □

2.3 Exponential Vectors, Convolution and the Operators $\mathcal{D}^c$ and $\mathcal{S}^c$

An important class of vectors from the Fock space $\mathcal{M}$ are the exponential vectors. They are used to model coherent beams.

Definition 2.12. *Given a function $h : G \longrightarrow \mathbb{C}$, the function $\mathbb{e}_h : M \longrightarrow \mathbb{C}$, defined by*

$$\mathbb{e}_h(\varphi) := \begin{cases} \prod_{x \in supp\, \varphi} h(x)^{\varphi(\{x\})} & \text{for } 0 < |\varphi| < \infty, \\ 1 & \text{for } \varphi = \mathbf{o}, \\ 0 & \text{else} \end{cases}$$

is called EXPONENTIAL VECTOR generated by h.

Observe that $\mathbb{e}_h \in \mathcal{M}$ if and only if $h \in \mathcal{L}^2(G,\nu)$. In this case there holds

$$\|\mathbb{e}_h\|^2 = \exp(\|h\|^2). \tag{2.3.1}$$

Moreover, the linear span of the exponential vectors from $\mathcal{M}$ is dense in $\mathcal{M}$.
So we can define bounded operators on $\mathcal{M}$ using only their restriction to the set of exponential vectors [37].
We now give some useful properties of exponential vectors.

Lemma 2.13. *Let f , $g : G \longrightarrow \mathbb{C}$ and $\varphi_1, \varphi_2, \varphi \in M$. Then*

$$\begin{aligned}
\mathbb{e}_f\ (\varphi_1+\varphi_2) &= \mathbb{e}_f\ (\varphi_1)\cdot \mathbb{e}_f\ (\varphi_2), && (2.3.2)\\
\mathbb{e}_{f+g}\ (\varphi) &= \sum_{\hat\varphi\subseteq\varphi} \binom{\varphi}{\hat\varphi} \mathbb{e}_f\ (\hat\varphi)\cdot \mathbb{e}_g\ (\varphi-\hat\varphi), && (2.3.3)\\
\mathbb{e}_{f\cdot g}\ (\varphi) &= \mathbb{e}_f\ (\varphi)\cdot \mathbb{e}_g\ (\varphi). && (2.3.4)
\end{aligned}$$

PROOF. This follows immedeately from Definition 2.12 and the Binomial Theorem. For more details see for example [30]. $\square$

We use exponential vectors to define operators of second quantization [37].

Definition 2.14. *Let $T \in \mathfrak{L}(\mathcal{L}^2(G,\nu))$ with $\|T\| \leq 1$.*
The uniquely determined bounded operator $\Gamma(T)$ on $\mathcal{M}$ with

$$\Gamma(T)\mathbb{e}_h = \mathbb{e}_{Th} \qquad (h \in \mathcal{L}^2(G,\nu)\ ) \tag{2.3.5}$$

is called SECOND QUANTIZATION of T.

Remark 2.15. *For S, $T \in \mathfrak{L}(\mathcal{L}^2(G,\nu))$ with $\|S\| \leq 1$, $\|T\| \leq 1$ there hold*

$$\Gamma(S)\Gamma(T) = \Gamma(ST) \tag{2.3.6}$$

and

$$\Gamma(T)^* = \Gamma(T^*). \tag{2.3.7}$$

For two arbitrary finite measures Q_1 and Q_2 on $[M, \mathfrak{M}]$ we define the CONVOLUTION $Q_1 * Q_2$:

$$(Q_1 * Q_2)(Y) := \int\limits_M Q_1(d\varphi_1) \int\limits_M Q_2(d\varphi_2)\chi_Y(\varphi_1+\varphi_2) \qquad (Y \in \mathfrak{M}).$$

For $n \geq 2$ measures the convolution is defined by induction:

$$\mathop{*}_{j=1}^{n} Q_j := (...(Q_1 * Q_2) * Q_3) * \ldots * Q_n).$$

We recall some properties of the convolution. For more details see [5], chapter 24.

Remark 2.16. *Let Q_1, Q_2, Q_3 be finite measures on $[M, \mathfrak{M}]$ and $a \in [0,\infty)$, $\varphi_1, \varphi_2 \in M$.*

(a) $Q_1 * (Q_2 + Q_3) = Q_1 * Q_2 + Q_1 * Q_3$,

(b) $(Q_1 * (a \cdot Q_2) = (a \cdot Q_1) * Q_2 = a \cdot (Q_1 * Q_2)$
(For point processes Q_1, Q_2 and $a \neq 1$ $\;a \cdot (Q_1 * Q_2)$ is not a point process but only a finite measure on $[M, \mathfrak{M}]$.),

(c) For Dirac measures δ_{φ_1} and δ_{φ_2} on $[M, \mathfrak{M}]$ there holds

$$\delta_{\varphi_1} * \delta_{\varphi_2} = \delta_{\varphi_1 + \varphi_2},$$

(d) $\delta_{\mathfrak{o}} * Q_1 = Q_1 * \delta_{\mathfrak{o}} = Q_1$.

Lemma 2.17. *Let k, $m \in \mathbb{N}$ and $a_1, \ldots, a_k \in [0,\infty)$, $\varphi_1, \ldots, \varphi_k \in M$. Then*

$$\left(a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k}\right)^{*m}$$

$$= \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=m}} \binom{m}{l_1}\binom{m-l_1}{l_2}\cdots\binom{m-\sum_{i=1}^{k-1} l_i}{l_k} a_1^{l_1}\ldots a_k^{l_k} \cdot \delta_{l_1\varphi_1+\ldots+l_k\varphi_k}. \quad (2.3.8)$$

In the special case $k = 2$ there follows

$$(a_1\delta_{\varphi_1} + a_2\delta_{\varphi_2})^{*m} = \sum_{l=0}^{m} \binom{m}{l} a_1^l a_2^{m-l} \cdot \delta_{l\varphi_1+(m-l)\varphi_2}.$$

PROOF. We will prove (2.3.8) by induction.
For $m = 1$ there is nothing to prove. For $m = 2$ we use the distributive law from Remark 2.16 and get

$$(a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k})^{*2} = \sum_{l,m=0}^{k} a_l a_m \delta_{\varphi_l+\varphi_m}$$

$$= \sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=2}} \binom{2}{l_1}\binom{2-l_1}{l_2}\cdots\binom{2-\sum_{i=1}^{k-1} l_i}{l_k} a_1^{l_1}\ldots a_k^{l_k}\; \delta_{l_1\varphi_1+\ldots+l_k\varphi_k}.$$

Assume that for $m \in \mathbb{N}$ (2.3.8) holds. For $m+1$ we get

$$(a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k})^{*(m+1)} = (a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k})^{*m} * (a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k})$$

$$= \left(\sum_{\substack{l_1,\ldots,l_k \geq 0 \\ l_1+\ldots+l_k=m}} \binom{m}{l_1}\binom{m-l_1}{l_2}\cdots\binom{m-\sum_{i=1}^{k-1} l_i}{l_k} a_1^{l_1}\ldots a_k^{l_k} \cdot \delta_{l_1\varphi_1+\ldots+l_k\varphi_k}\right) * (a_1\delta_{\varphi_1} + \ldots + a_k\delta_{\varphi_k})$$

$$= \sum_{s=1}^{k} \Big(\sum_{\substack{l_1,\dots,l_k \geq 0 \\ l_1+\dots+l_k=m}} \binom{m}{l_1} \binom{m-l_1}{l_2} \dots \binom{m-\sum_{i=1}^{k-1} l_i}{l_k} a_1^{l_1} \dots a_k^{l_k} a_s \cdot \delta_{l_1\varphi_1+\dots+l_k\varphi_k+\varphi_s} \Big)$$

$$= \sum_{\substack{l_1,\dots,l_k \geq 0 \\ l_1+\dots+l_k=m+1}} \binom{m+1}{l_1} \binom{m+1-l_1}{l_2} \dots \binom{m+1-\sum_{i=1}^{k-1} l_i}{l_k} a_1^{l_1} \dots a_k^{l_k} \cdot \delta_{l_1\varphi_1+\dots+l_k\varphi_k}.$$

□

Now we introduce the compound Malliavin derivative and the compound Skorohod integral.

Definition 2.18. *The operator $\mathcal{D}^c : dom\mathcal{D}^c \to \mathcal{M}^2$ given on the domain $dom\mathcal{D}^c := \{\Psi \in \mathcal{M} : \int F(d\varphi)|\Psi(\varphi)|^2 2^{|\varphi|} < \infty\}$ by*

$$\mathcal{D}^c\Psi(\varphi_1, \varphi_2) := \Psi(\varphi_1 + \varphi_2) \qquad (\Psi \in dom\mathcal{D}^c, \varphi_1, \varphi_2 \in M) \tag{2.3.9}$$

is called COMPOUND MALLIAVIN DERIVATIVE.
The operator $\mathcal{S}^c : dom\mathcal{S}^c \to \mathcal{M}$ given on the domain $dom\mathcal{S}^c := \{\Phi \in \mathcal{M}^2 : \int F(d\varphi_1) \int F(d\varphi_2)|\Phi(\varphi_1, \varphi_2)|^2 \cdot 2^{|\varphi_1|+|\varphi_2|} < \infty\}$ by

$$\mathcal{S}^c\Phi(\varphi) := \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} \Phi(\hat{\varphi}, \varphi - \hat{\varphi}) \qquad (\Phi \in dom\mathcal{S}^c, \varphi \in M) \tag{2.3.10}$$

is called COMPOUND SKOROHOD INTEGRAL.

Lemma 2.19. *$dom\mathcal{D}^c$ given in Definition 2.18 is the maximal domain of definition of $\mathcal{D}^c$.*

PROOF. Let $\Psi \in dom\mathcal{D}^c$. Using part (b) of Proposition 2.11 and Lemma 2.13 we get

$$\begin{aligned}
\|\mathcal{D}^c\Psi\|^2_{\mathcal{M}^2} &= \int F(d\varphi_1) \int F(d\varphi_2)|\mathcal{D}^c\Psi(\varphi_1, \varphi_2)|^2 = \int F(d\varphi_1) \int F(d\varphi_2)|\Psi(\varphi_1 + \varphi_2)|^2 \\
&= \int F(d\varphi) \sum_{\widehat{\varphi}\subseteq\varphi} \binom{\varphi}{\widehat{\varphi}} |\Psi(\varphi)|^2 = \int F(d\varphi)|\Psi(\varphi)|^2 \sum_{\widehat{\varphi}\subseteq\varphi} \binom{\varphi}{\widehat{\varphi}} \mathbb{e}_1(\widehat{\varphi}) \cdot \mathbb{e}_1(\varphi - \widehat{\varphi}) \\
&= \int F(d\varphi)|\Psi(\varphi)|^2 \mathbb{e}_2(\varphi) = \int F(d\varphi)|\Psi(\varphi)|^2 \cdot 2^{|\varphi|}.
\end{aligned}$$

□

Corollary 2.20. *For all functions $g \in \mathcal{L}^2(G, \nu)$ it holds $\mathbb{e}_g \in dom\mathcal{D}^c$.*

PROOF. From Lemma 2.19 and (2.3.1) there follows

$$\|\mathcal{D}^c \mathbb{e}_g\|^2 = \int F(d\varphi)|\mathbb{e}_g(\varphi)|^2 \cdot 2^{|\varphi|} = \int F(d\varphi)|\mathbb{e}_{\sqrt{2}g}(\varphi)|^2 = \|\mathbb{e}_{\sqrt{2}g}\|^2 = e^{2\|g\|^2} < \infty.$$

□

Corollary 2.21. *For all functions g, $h \in \mathcal{L}^2(G,\nu)$ it holds $\mathbb{e}_g \otimes \mathbb{e}_h \in dom\mathcal{S}^c$.*

PROOF. From Corollary 2.20 there follows

$$\begin{aligned} &\int F(d\varphi_1) \int F(d\varphi_2)|\mathbb{e}_g(\varphi_1) \cdot \mathbb{e}_h(\varphi_2)|^2 \cdot 2^{|\varphi_1|+|\varphi_2|} \\ &\quad = \left(\int F(d\varphi_1)|\mathbb{e}_g(\varphi_1)|^2 \cdot 2^{|\varphi_1|}\right) \cdot \left(\int F(d\varphi_2)|\mathbb{e}_h(\varphi_2)|^2 \cdot 2^{|\varphi_2|}\right) \\ &\quad = \|\mathcal{D}^c \mathbb{e}_g\|^2_{\mathcal{M}^2} \cdot \|\mathcal{D}^c \mathbb{e}_h\|^2_{\mathcal{M}^2} < \infty. \end{aligned}$$

□

Remark 2.22. *$\mathcal{D}^c$ and $\mathcal{S}^c$ are unbounded operators. For all $\Psi \in dom\mathcal{D}^c$ and all $\Phi \in dom\mathcal{S}^c$ we get*

$$\begin{aligned} \langle \mathcal{D}^c\Psi, \Phi\rangle_{\mathcal{M}^2} &= \int F(d\varphi_2) \int F(d\varphi_1)\overline{\mathcal{D}^c\Psi(\varphi_1,\varphi_2)}\Phi(\varphi_1,\varphi_2) \\ &= \int F(d\varphi_2) \int F(d\varphi_1)\overline{\Psi(\varphi_1+\varphi_2)}\Phi(\varphi_1,\varphi_2) \\ &= \int F(d\varphi) \sum_{\hat\varphi\subseteq\varphi} \binom{\varphi}{\hat\varphi} \overline{\Psi(\hat\varphi+\varphi-\hat\varphi)}\Phi(\hat\varphi,\varphi-\hat\varphi) \\ &= \int F(d\varphi)\overline{\Psi(\varphi)} \sum_{\hat\varphi\subseteq\varphi} \binom{\varphi}{\hat\varphi} \Phi(\hat\varphi,\varphi-\hat\varphi) = \int F(d\varphi)\overline{\Psi(\varphi)}\mathcal{S}^c\Phi(\hat\varphi,\varphi-\hat\varphi) \\ &= \langle \Psi, \mathcal{S}^c\Phi\rangle_{\mathcal{M}}, \end{aligned}$$

hence, $\mathcal{D}^c$ and $\mathcal{S}^c$ are mutually adjoint.
On exponential vectors $\mathbb{e}_f$, $\mathbb{e}_g$ with f,$g \in \mathcal{L}^2(G,\nu)$ we get immediately from Lemma 2.13

$$\mathcal{D}^c\mathbb{e}_g = \mathbb{e}_g \otimes \mathbb{e}_g \quad \textit{and} \tag{2.3.11}$$
$$\mathcal{S}^c(\mathbb{e}_f \otimes \mathbb{e}_g) = \mathbb{e}_{f+g}. \tag{2.3.12}$$

Remark 2.23. *We get the Malliavin derivative $\mathcal{D}$ on the Fock space from $\mathcal{D}^c$ by restricting the first variable to one-point configurations, i.e.*

$$\mathcal{D}\Psi(x,\varphi) = \mathcal{D}^c\Psi(\delta_x,\varphi) \qquad (\Psi \in \mathcal{M},\ x \in G,\ \varphi \in M).$$

The adjoint of $\mathcal{D}$ is described by

$$\mathcal{S}\Phi(\varphi) = \int \varphi(dx)\varphi(\{x\})\Phi(x,\varphi-\delta_x).$$

For more details on Malliavin derivative and Skorohod integral see for instance [24, 18, 12].

Proposition 2.24. *Let $n \in \mathbb{N}$ and the mapping $(\mathcal{D}^c)^n : dom(\mathcal{D}^c)^n \longrightarrow \mathcal{M}^{n+1}$ be defined on the domain*
$dom(\mathcal{D}^c)^n = \{\Psi \in \mathcal{M}^n : \int F^n(d[\varphi_1, \ldots, \varphi_n])|\Psi(\varphi_1, \ldots \varphi_n)|^2 \cdot 2^{|\varphi_1 + \ldots + \varphi_n|} < \infty\}$ by

$$((\mathcal{D}^c)^n\Psi)(\varphi_0, \ldots, \varphi_n) := \Psi(\varphi_0 + \ldots + \varphi_n) \quad (\Psi \in dom(\mathcal{D}^c)^n,\ \varphi_0, \ldots, \varphi_n \in M). \tag{2.3.13}$$

Then $(\mathcal{D}^c)^n$ can be expressed recursively by

$$\begin{aligned} (\mathcal{D}^c)^n &= (\mathbb{1} \otimes (\mathcal{D}^c)^{n-1})\mathcal{D}^c = (\mathbb{1}^{n-1} \otimes \mathcal{D}^c)(\mathcal{D}^c)^{n-1} \quad (n \geq 2) && (2.3.14)\\ (\mathcal{D}^c)^1 &= \mathcal{D}^c. && (2.3.15) \end{aligned}$$

where $\mathbb{1} := \mathbb{1}_{\mathfrak{L}(\mathcal{M})}$.

PROOF. For $n \in \mathbb{N}$, $n \geq 2$, $\Psi \in \mathcal{M}$ and $\varphi_0, \ldots, \varphi_n \in M$ we get

$$\begin{aligned} &(\mathbb{1} \otimes (\mathcal{D}^c)^{n-1})\mathcal{D}^c\Psi(\varphi_0, \ldots, \varphi_n) = (\mathcal{D}^c)^{n-1}(\mathcal{D}^c\Psi(\varphi_0, .))(\varphi_1, \ldots, \varphi_n)\\ &\quad = \mathcal{D}^c\Psi(\varphi_0, \varphi_1 + \ldots + \varphi_n) = \Psi(\varphi_0 + \ldots + \varphi_n) = (\mathcal{D}^c)^n\Psi(\varphi_0, \ldots, \varphi_n) \end{aligned}$$

and

$$\begin{aligned} &(\mathbb{1}^{n-1} \otimes \mathcal{D}^c)(\mathcal{D}^c)^{n-1}\Psi(\varphi_0, \ldots, \varphi_n) = \mathcal{D}^c((\mathcal{D}^c)^{n-1}\Psi(\varphi_0, \ldots, \varphi_{n-1}))(., \varphi_n)\\ &\quad = \mathcal{D}^c\Psi(\varphi_0 + \ldots + \varphi_{n-1}, \varphi_n) = \Psi(\varphi_0 + \ldots + \varphi_n) = (\mathcal{D}^c)^n\Psi(\varphi_0, \ldots, \varphi_n). \end{aligned}$$

□

Remark 2.25. *From Definition 2.18, Lemma 2.19 and Proposition 2.24 we see that*

$$dom(\mathcal{D}^c)^n = \{\Psi \in \mathcal{M} : \int F(d\varphi)|\Psi(\varphi)|^2 \cdot (n+1)^{|\varphi|} < \infty\} \tag{2.3.16}$$

is the maximal domain of definition for $(\mathcal{D}^c)^n$.
On exponential vectors $\mathbb{e}_h$ with $h \in \mathcal{L}^2(G, \nu)$ it holds $(\mathcal{D}^c)^n\mathbb{e}_h = (\mathbb{e}_h)^{\otimes(n+1)}$.

Proposition 2.26. *Let $n \in \mathbb{N}$ and the mapping $(\mathcal{S}^c)^n : dom(\mathcal{S}^c)^n \longrightarrow \mathcal{M}$ be defined for $\Phi \in dom(\mathcal{S}^c)^n$ and $\varphi \in M$ by*

$$((\mathcal{S}^c)^n\Phi)(\varphi) := \sum_{\varphi_0 + \ldots + \varphi_n = \varphi} \prod_{k=0}^{n} \binom{\varphi_0 + \ldots + \varphi_{k+1}}{\varphi_0 + \ldots + \varphi_k} \Phi(\varphi_0, \ldots, \varphi_n) \tag{2.3.17}$$

$$= \sum_{\varphi_0 \subseteq \ldots \subseteq \varphi_n = \varphi} \binom{\varphi_1}{\varphi_0} \cdots \binom{\varphi_n}{\varphi_{n-1}} \Phi(\varphi_0, \varphi_1 - \varphi_0, \ldots, \varphi_n - \varphi_{n-1}) \tag{2.3.18}$$

with

$$dom(\mathcal{S}^c)^n := \{\Phi \in \mathcal{M}^{n+1} : \int F^{n+1}(d\underline{\varphi})|\Phi(\underline{\varphi})|^2 \cdot (n+1)^{|\varphi_1| + \ldots + |\varphi_{n+1}|} < \infty\}, \tag{2.3.19}$$

where $\underline{\varphi} := (\varphi_1, \ldots, \varphi_{n+1})$ *with* $\varphi_k \in M$ *for* $k \in (n+1)]$.
Then $(\mathcal{D}^c)^n$ *and* $(\mathcal{S}^c)^n$ *are mutually adjoint.*
$(\mathcal{S}^c)^n$ *can be expressed recursively by*

$$\begin{aligned}(\mathcal{S}^c)^n &= \mathcal{S}^c(\mathbb{1}\otimes(\mathcal{S}^c)^{n-1}) = (\mathcal{S}^c)^{n-1}(\mathbb{1}^{n-1}\otimes\mathcal{S}^c) \quad (n\geq 2) &(2.3.20)\\ (\mathcal{S}^c)^1 &= \mathcal{S}^c. &(2.3.21)\end{aligned}$$

where $\mathbb{1} := \mathbb{1}_{\mathfrak{L}(\mathcal{M})}$.

PROOF. Let $\Psi \in dom(\mathcal{D}^c)^n$ and $\Phi \in dom(\mathcal{S}^c)^n$. Using part (d) of Proposition 2.11 we get

$$\langle(\mathcal{D}^c)^n\Psi,\Phi\rangle_{\mathcal{M}^{n+1}} = \int F(d\varphi_1)\ldots\int F(d\varphi_{n+1})\overline{(\mathcal{D}^c)^n\Psi}(\varphi_1,\ldots,\varphi_{n+1})\Phi(\varphi_1,\ldots,\varphi_{n+1})$$
$$= \int F(d\varphi_1)\ldots\int F(d\varphi_{n+1})\overline{\Psi(\varphi_1+\ldots+\varphi_{n+1})}\Phi(\varphi_1,\ldots,\varphi_{n+1})$$
$$= \int F(d\varphi_{n+1})\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n+1}}\binom{\varphi_2}{\varphi_1}\ldots\binom{\varphi_{n+1}}{\varphi_n}\overline{\Psi(\varphi_{n+1})}\Phi(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_{n+1}-\varphi_n)$$
$$= \int F(d\varphi_{n+1})\overline{\Psi(\varphi_{n+1})}\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n+1}}\binom{\varphi_2}{\varphi_1}\ldots\binom{\varphi_{n+1}}{\varphi_n}\Phi(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_{n+1}-\varphi_n)$$
$$= \int F(d\varphi)\overline{\Psi(\varphi)}(\mathcal{S}^c)^n\Phi(\varphi) = \langle\Psi,(\mathcal{S}^c)^n\Phi\rangle_{\mathcal{M}}.$$

Hence, $(\mathcal{D}^c)^n$ and $(\mathcal{S}^c)^n$ are mutually adjoint.
From this and (2.3.14) there follows for all $n \geq 2$

$$(\mathcal{S}^c)^n = [(\mathcal{D}^c)^n]^* = [(\mathbb{1}\otimes(\mathcal{D}^c)^{n-1})\mathcal{D}^c]^* = (\mathcal{D}^c)^*[(\mathbb{1}\otimes((\mathcal{D}^c)^{n-1})^*] = \mathcal{S}^c(\mathbb{1}\otimes(\mathcal{S}^c)^{n-1})$$

and

$$\begin{aligned}(\mathcal{S}^c)^n &= [(\mathcal{D}^c)^n]^* = [(\mathbb{1}^{n-1}\otimes\mathcal{D}^c)(\mathcal{D}^c)^{n-1}]^* = [(\mathcal{D}^c)^{n-1}]^*(\mathbb{1}^{n-1}\otimes(\mathcal{D}^c)^*)\\ &= (\mathcal{S}^c)^{n-1}(\mathbb{1}^{n-1}\otimes\mathcal{S}^c).\end{aligned}$$

□

Remark 2.27. *From Definition 2.18, Corollary 2.21 and Proposition 2.26 we see that for all* $n \in \mathbb{N}$ *tensor products* $\mathbb{e}_{h_1}\otimes\cdots\otimes\mathbb{e}_{h_{n+1}}$ *of exponential vectors with* $h_1,\ldots,h_{n+1}$ *from* $\mathcal{L}^2(G,\nu)$ *are contained in* $dom(\mathcal{S}^c)^n$. *In this case it holds* $(\mathcal{S}^c)^n\mathbb{e}_{h_1}\otimes\ldots\otimes\mathbb{e}_{h_{n+1}} = \mathbb{e}_{h_1+\ldots+h_{n+1}}$.

Definition 2.28. *Let* $n \in \mathbb{N}$. *For functions* $f : M^n \longrightarrow \mathbb{C}$ *we denote by* O_f *the operator of multiplication by* f, *i.e.*

$$O_f\Phi(\underline{\varphi}) := f(\underline{\varphi})\cdot\Phi(\underline{\varphi}) \qquad (\Phi\in\mathcal{M}^n,\ \underline{\varphi}\in M^n) \tag{2.3.22}$$

where $M^n := \{\underline{\varphi} = (\varphi_1,\ldots,\varphi_n)\ ;\ \varphi_i \in M\ \forall i \in n]\}$ *denotes the set of all* n-*dimensional vectors with components from* M.
In the case where $f = \chi_{\underline{Y}}$ *is the indicator function of a set* $\underline{Y} \in \mathfrak{M}^n$ *we will write* $O_{\underline{Y}}$ *instead of* $O_{\chi_{\underline{Y}}}$.

Chapter 3

Generalized Splitting Procedures

3.1 Definition and Basic Properties

In this chapter we will consider transition expectations and corresponding quantum Markov chains resulting from generalizations of beam splitting procedures.
The initial point for the investigation of such quantum Markov chains were those originating from independent beam splittings [24, 26, 19]. Later this model was expanded to dependent splittings [27, 32].
Now we consider general interaction procedures with one input and two outputs in every step. These procedures in general are no beam splittings, but in consideration of the history of origin we will call them generalized splitting procedures.
We will define transition expectations $\mathcal{E}$ on $\mathcal{B} \otimes \mathcal{A}$ with $\mathcal{A} = \mathcal{B} = \mathfrak{L}(\mathcal{M})$. To distinguish better between the quantum system and the measurement apparatus, we will still use the notations $\mathcal{A}$ and $\mathcal{B}$ for the algebras of observables of the quantum system and the measurement apparatus, respectively.

First we define the bounded operator $\mathcal{V}$ from $\mathcal{M}$ to $\mathcal{M}^2$ that plays a basic role in the definition of the transition expectations. For this we need a so-called SPLITTING FUNCTION g.

Let $g : M \times M \longrightarrow \mathbb{C}$ be a function satisfying

$$\sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} |g(\widehat{\varphi}, \varphi - \widehat{\varphi})|^2 = 1 \quad \text{for } F\text{-almost all } \varphi \in M. \tag{3.1.1}$$

We define the bounded operator $\mathcal{V} : \mathcal{M} \longrightarrow \mathcal{M}^2$ on $\mathcal{M}$ by

$$\mathcal{V}\ \Psi(\varphi_1, \varphi_2) := g(\varphi_1, \varphi_2) \cdot \Psi(\varphi_1 + \varphi_2) \qquad (\Psi \in \mathcal{M}\ ,\ \varphi_1, \varphi_2 \in M). \tag{3.1.2}$$

With the multiplication operator O_g given in Definition 2.28 and the compound Malliavin derivative defined in (2.3.9) we can write

$$\mathcal{V} = O_g \mathcal{D}^c. \tag{3.1.3}$$

Lemma 3.1. *Let $\mathcal{V}$ be defined according to* (3.1.2). *Then*

(a) *$\mathcal{V}$ is an isometry iff* (3.1.1) *holds.*

(b) *Let* (3.1.1) *be fulfilled. Then the adjoint operator $\mathcal{V}^* : \mathcal{M}^2 \longrightarrow \mathcal{M}$ of $\mathcal{V}$ is given by*

$$\mathcal{V}^*\Phi(\varphi) = \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} \overline{g}(\hat{\varphi}, \varphi - \hat{\varphi})\Phi(\hat{\varphi}, \varphi - \hat{\varphi}) \qquad (\Phi \in \mathcal{M}^2, \ \varphi \in M) \quad (3.1.4)$$

with $\overline{g}$ denoting the complex conjugate function of g.

PROOF. (a) Let $\Psi \in \mathcal{M}$. Then, according to Proposition 2.11,

$$\begin{aligned} \|\mathcal{V}\Psi\|^2_{\mathcal{M}^2} &= \int F(d\varphi_2) \int F(d\varphi_1)|g(\varphi_1, \varphi_2)|^2 |\Psi(\varphi_1 + \varphi_2)|^2 \\ &= \int F(d\varphi) \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} |g(\hat{\varphi}, \varphi - \hat{\varphi})|^2 |\Psi(\varphi)|^2. \end{aligned} \qquad (3.1.5)$$

" $\Longleftarrow$ " : If (3.1.1) is valid then we have $\|\Psi\|^2_{\mathcal{M}} = \int F(d\varphi)|\Psi(\varphi)|^2 = \|\mathcal{V}\Psi\|^2_{\mathcal{M}^2}$ for all $\Psi \in \mathcal{M}$.
" $\Longrightarrow$ " : Let $\|\mathcal{V}\Psi\|^2_{\mathcal{M}^2} = \|\Psi\|^2_{\mathcal{M}}$ for all $\Psi \in \mathcal{M}$. Because of (3.1.5) this implies (3.1.1).
(b) Let $\Psi \in \mathcal{M}$ and $\Phi \in \mathcal{M}^2$. Then, applying Proposition 2.11, we get

$$\begin{aligned} \langle \Phi, \mathcal{V}\Psi \rangle_{\mathcal{M}^2} &= \int F(d\varphi_2) \int F(d\varphi_1) \overline{\Phi(\varphi_1, \varphi_2)} g(\varphi_1, \varphi_2) \Psi(\varphi_1 + \varphi_2) \\ &= \int F(d\varphi) \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} \overline{\Phi(\hat{\varphi}, \varphi - \hat{\varphi})} g(\hat{\varphi}, \varphi - \hat{\varphi}) \Psi(\varphi) \\ &= \langle \mathcal{V}^*\Phi, \Psi \rangle_{\mathcal{M}}, \end{aligned}$$

hence, $\mathcal{V}^*$ is the adjoint operator of $\mathcal{V}$. □

Now we want to use $\mathcal{V}$ to define a transition expectation (as it was given in Definition 1.2).
Remember that $\mathcal{A} = \mathcal{B} = \mathfrak{L}(\mathcal{M})$.
Because $\mathcal{V}$ is an isometry, by

$$\mathcal{E}(B \otimes A) := \mathcal{V}^*(B \otimes A)\mathcal{V}$$

there can be defined a transition expectation $\mathcal{E} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ (as it was done in [24]).
Except for the interaction of the quantum system and the measurement apparatus the whole system may undergo an additional evolution. In the most general case the generalized splitting procedure could be described by $\mathcal{E} \circ W$ where W is a mapping on $\mathcal{B} \otimes \mathcal{A}$ characterizing the additional transformation. More specific, W could be of the type $W = \tau_U = U^*(\cdot)U$ with $U \in \mathcal{B} \otimes \mathcal{A}$ and $U = U_1 \otimes U_2$. U_1 and U_2 are

isometric operators on $\mathcal{M}$ describing independent evolutions of the measurement apparatus and the quantum system, respectively.
System evolutions were considered in [19].

Now we define a new transition expectation $\mathcal{E}_{U_1,U_2}$ taking into account additional transformations of the quantum system and the measurement apparatus.
Let $\mathcal{E}_{U_1,U_2} : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be defined by

$$\mathcal{E}_{U_1,U_2}(B \otimes A) := \mathcal{V}^*(U_1^* B U_1 \otimes U_2^* A U_2)\mathcal{V} \qquad (A \in \mathcal{A}\ ,\ B \in \mathcal{B}), \tag{3.1.6}$$

where U_1 and U_2 are isometric operators on $\mathcal{M}$. This ensures that $\mathcal{E}_{U_1,U_2}$ is again completely positive and identity preserving:

Proposition 3.2. *The mapping $\mathcal{E}_{U_1,U_2}$ defined by* (3.1.6) *with U_1 and U_2 being isometric operators on $\mathcal{M}$ is an isometric transition expectation of the form*

$$\mathcal{E}_{U_1,U_2} = (\mathcal{V}_{U_1,U_2})^*(\cdot)\mathcal{V}_{U_1,U_2}$$

where

$$\mathcal{V}_{U_1,U_2} := (U_1 \otimes U_2)\mathcal{V} \quad \textit{and} \quad (\mathcal{V}_{U_1,U_2})^* := \mathcal{V}^*(U_1^* \otimes U_2^*).$$

PROOF. Rewriting the definition of $\mathcal{E}_{U_1,U_2}$ we get for $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\mathcal{E}_{U_1,U_2}(B \otimes A) := \mathcal{V}^*(U_1^* \otimes U_2^*)(B \otimes A)(U_1 \otimes U_2)\mathcal{V}.$$

Consequently, $\mathcal{E}_{U_1,U_2}$ has the described form. From Lemma 3.1 we know that $\mathcal{V}$ is an isometry. Because U_1 and U_2 are assumed to be isometric, this implies $(\mathcal{V}_{U_1,U_2})^*\mathcal{V}_{U_1,U_2} = \mathbb{1}_{\mathcal{A}}$. Because of Lemma 1.4, $\mathcal{E}_{U_1,U_2}$ is a transition expectation.
□

Remark 3.3. *Using the compound Malliavin derivative and compound Skorohod integral given in Definition 2.18, we can write*

$$\mathcal{V}_{U_1,U_2} = (U_1 \otimes U_2) O_g \mathcal{D}^c, \tag{3.1.7}$$
$$(\mathcal{V}_{U_1,U_2})^* = \mathcal{S}^c O_{\overline{g}}(U_1^* \otimes U_2^*) \tag{3.1.8}$$

where O_g is the operator of multiplication by the splitting function g.

Now we consider a sequence generalized splitting procedures according to the same splitting rule. Let again $\mathcal{A} = \mathcal{B} = \mathfrak{L}(\mathcal{M})$.

Proposition 3.4. *Let $n \in \mathbb{N}$, $A \in \mathcal{A}$, $B_1, \dots B_n \in \mathcal{B}$ and U_1, U_2 isometric operators on $\mathcal{M}$. The mapping $\mathcal{E}^n_{U_1,U_2} : \mathcal{B}^n \otimes \mathcal{A} \longrightarrow \mathcal{A}$ given by*

$$\begin{aligned} \mathcal{E}^n_{U_1,U_2}(B_1 \otimes \ldots \otimes B_n \otimes A) &:= \mathcal{E}_{U_1,U_2}(B_1 \otimes \mathcal{E}^{n-1}_{U_1,U_2}(B_2 \otimes \ldots \otimes B_n \otimes A)) \qquad (n \geq 2) \\ \mathcal{E}^1_{U_1,U_2} &:= \mathcal{E}_{U_1,U_2} \end{aligned}$$

is an isometric transition expectation.
$\mathcal{E}^n_{U_1,U_2}$ *is of the form*

$$\mathcal{E}^n_{U_1,U_2} = (\mathcal{V}^n_{U_1,U_2})^* \ (\ \cdot \ ) \ \mathcal{V}^n_{U_1,U_2},$$

the isometry $\mathcal{V}^n_{U_1,U_2} : \mathcal{M} \longrightarrow \mathcal{M}^{\otimes(n+1)}$ *is given inductively by*

$$\mathcal{V}^n_{U_1,U_2} = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}^{n-1}_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \quad (n \geq 2) \tag{3.1.9}$$

$$\mathcal{V}^1_{U_1,U_2} = (U_1 \otimes U_2)\mathcal{V}. \tag{3.1.10}$$

Moreover, for $n \geq 2$ $\mathcal{V}^n_{U_1,U_2}$ *has the explicit form*

$$\mathcal{V}^n_{U_1,U_2} = (\mathbb{1}^{(n-1)}_{\mathcal{B}} \otimes \mathcal{V}_{U,1})\mathcal{V}_{U,n-1} \tag{3.1.11}$$

$$= \prod_{k=0}^{n-1} (\mathbb{1}^{(n-k-1)}_{\mathcal{B}} \otimes U_1 \otimes U_2)(\mathbb{1}^{(n-k-1)}_{\mathcal{B}} \otimes \mathcal{V}). \tag{3.1.12}$$

PROOF. First, let $n = 2$.

$$\begin{aligned}
\mathcal{E}^2_{U_1,U_2}(B_1 \otimes B_2 \otimes A) &= \mathcal{E}_{U_1,U_2}(B_1 \otimes \mathcal{E}_{U_1,U_2}(B_2 \otimes A)) \\
&= (\mathcal{V}^1_{U_1,U_2})^*(B_1 \otimes (\mathcal{V}^1_{U_1,U_2})^*(B_2 \otimes A)\mathcal{V}^1_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^1_{U_1,U_2})^*(\mathbb{1}_{\mathcal{B}} \otimes (\mathcal{V}^1_{U_1,U_2})^*)(B_1 \otimes B_2 \otimes A)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}^1_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^2_{U_1,U_2})^*(B_1 \otimes B_2 \otimes A)\mathcal{V}^2_{U_1,U_2}
\end{aligned} \tag{3.1.13}$$

Now for $n \in \mathbb{N}$ let $\mathcal{E}^n_{U_1,U_2} = (\mathcal{V}^n_{U_1,U_2})^* \ (\ \cdot \ ) \ \mathcal{V}^n_{U_1,U_2}$. Then

$$\begin{aligned}
&\mathcal{E}^{n+1}_{U_1,U_2}(B_1 \otimes \ldots \otimes B_{n+1} \otimes A) = \mathcal{E}_{U_1,U_2}(B_1 \otimes \mathcal{E}^n_{U_1,U_2}(B_2 \otimes \ldots \otimes B_{n+1} \otimes A)) \\
&= \mathcal{E}_{U_1,U_2}\left(B_1 \otimes (\mathcal{V}^n_{U_1,U_2})^*(B_2 \otimes \ldots \otimes B_{n+1} \otimes A)\mathcal{V}^n_{U_1,U_2} \right) \\
&= (\mathcal{V}^1_{U_1,U_2})^*\left(B_1 \otimes (\mathcal{V}^n_{U_1,U_2})^*(B_2 \otimes \ldots \otimes B_{n+1} \otimes A)\mathcal{V}^n_{U_1,U_2} \right)\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^1_{U_1,U_2})^*(\mathbb{1}_{\mathcal{B}} \otimes (\mathcal{V}^n_{U_1,U_2})^*)(B_1 \otimes \ldots \otimes B_{n+1} \otimes A)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}^n_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^{n+1}_{U_1,U_2})^*(B_1 \otimes \ldots \otimes B_{n+1} \otimes A)\mathcal{V}^{n+1}_{U_1,U_2}.
\end{aligned}$$

Now we want to prove (3.1.11) by induction. For $n = 2$ the result follows from (3.1.13). Furthermore, for $n > 2$

$$\begin{aligned}
&\mathcal{E}^{n+1}_{U_1,U_2}(B_1 \otimes \ldots \otimes B_{n+1} \otimes A) = \mathcal{E}_{U_1,U_2}(B_1 \otimes \mathcal{E}^n_{U_1,U_2}(B_2 \otimes \ldots \otimes B_{n+1} \otimes A)) \\
&= (\mathcal{V}^1_{U_1,U_2})^*(B_1 \otimes (\mathcal{V}^n_{U_1,U_2})^*(B_2 \otimes \ldots \otimes B_{n+1} \otimes A)\mathcal{V}^n_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^1_{U_1,U_2})^*(B_1 \otimes (\mathcal{V}^{n-1}_{U_1,U_2})^*(\mathbb{1}^{n-1}_{\mathcal{B}} \otimes (\mathcal{V}^1_{U_1,U_2})^*) \\
&\qquad\qquad (B_2 \otimes \ldots \otimes B_{n+1} \otimes A)(\mathbb{1}^{n-1}_{\mathcal{B}} \otimes \mathcal{V}^1_{U_1,U_2})\mathcal{V}^{n-1}_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^1_{U_1,U_2})^*(\mathbb{1}_{\mathcal{B}} \otimes (\mathcal{V}^{n-1}_{U_1,U_2})^*(\mathbb{1}^{n-1}_{\mathcal{B}} \otimes (\mathcal{V}^1_{U_1,U_2})^*) \\
&\qquad\qquad (B_1 \otimes \ldots \otimes B_{n+1} \otimes A)(\mathbb{1}_{\mathcal{B}} \otimes (\mathbb{1}^{n-1}_{\mathcal{B}} \otimes \mathcal{V}^1_{U_1,U_2})(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}^{n-1}_{U_1,U_2})))\mathcal{V}^1_{U_1,U_2} \\
&= (\mathcal{V}^1_{U_1,U_2})^*(\mathbb{1}_{\mathcal{B}} \otimes (\mathcal{V}^{n-1}_{U_1,U_2})^*)(\mathbb{1}^n_{\mathcal{B}} \otimes (\mathcal{V}^1_{U_1,U_2})^*)
\end{aligned}$$

$$(B_1 \otimes \ldots \otimes B_{n+1} \otimes A)(\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V}_{U_1,U_2}^1)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_{U_1,U_2}^{n-1})\mathcal{V}_{U_1,U_2}^1$$
$$= (\mathcal{V}_{U_1,U_2}^n)^*(\mathbb{1}_{\mathcal{B}}^n \otimes (\mathcal{V}_{U_1,U_2}^1)^*)(B_1 \otimes \ldots \otimes B_{n+1} \otimes A)(\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V}_{U_1,U_2}^1)\mathcal{V}_{U_1,U_2}^n.$$

Hence,

$$\mathcal{V}_{U_1,U_2}^{n+1} = (\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V}_{U_1,U_2}^1)\mathcal{V}_{U_1,U_2}^n = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_{U_1,U_2}^n)\mathcal{V}_{U_1,U_2}^1.$$

Now we prove (3.1.12). First let $n = 2$.

$$\mathcal{V}_{U_1,U_2}^2 = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_{U_1,U_2}^1)\mathcal{V}_{U_1,U_2}^1 = (\mathbb{1}_{\mathcal{B}} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V})(U_1 \otimes U_2)\mathcal{V}.$$

Now for $n \in \mathbb{N}$ suppose $\mathcal{V}_{U_1,U_2}^n = \prod_{k=0}^{n-1}(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes \mathcal{V})$. Then

$$\mathcal{V}_{U_1,U_2}^{n+1} = (\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V}_{U_1,U_2}^1)\mathcal{V}_{U_1,U_2}^n = (\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V}_{U_1,U_2}^1)\prod_{k=0}^{n-1}(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes \mathcal{V})$$
$$= (\mathbb{1}_{\mathcal{B}}^n \otimes (U_1 \otimes U_2)\mathcal{V})\prod_{k=0}^{n-1}(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes \mathcal{V})$$
$$= (\mathbb{1}_{\mathcal{B}}^n \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V})\prod_{k=0}^{n-1}(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k-1)} \otimes \mathcal{V})$$
$$= (\mathbb{1}_{\mathcal{B}}^n \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^n \otimes \mathcal{V})\prod_{k=1}^{n}(\mathbb{1}_{\mathcal{B}}^{(n-k)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k)} \otimes \mathcal{V})$$
$$= \prod_{k=0}^{n}(\mathbb{1}_{\mathcal{B}}^{(n-k)} \otimes U_1 \otimes U_2)(\mathbb{1}_{\mathcal{B}}^{(n-k)} \otimes \mathcal{V}).$$

Because $\mathcal{V}_{U_1,U_2}$ is an isometry this holds also for all $\mathcal{V}_{U_1,U_2}^n$. Lemma 1.4 implies that $\mathcal{E}_{U_1,U_2}^n$ is a transition expectation.

□

Corollary 3.5. *Let $U_1 := \mathbb{1}_{\mathcal{B}}$, $U_2 := \mathbb{1}_{\mathcal{A}}$ and $\mathcal{E}^n := \mathcal{E}_{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}^n$ be defined as in Proposition 3.4. Then the isometric transition expectation $\mathcal{E}_{U_1,U_2}^n$ is of the form*

$$\mathcal{E}_{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}^n = \mathcal{E}^n = \mathcal{V}_n^*(\cdot)\mathcal{V}_n$$

with $\mathcal{V}_1 = \mathcal{V}$ and for $n \geq 2$

$$\mathcal{V}_n = \mathcal{V}_{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}^n = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_{n-1})\mathcal{V} = \prod_{k=0}^{n-1}(\mathbb{1}_{\mathcal{B}}^{n-k-1} \otimes \mathcal{V}).$$

PROOF. This follows immediately from Proposition 3.4. □

Proposition 3.6. *Let for $n \in \mathbb{N}$ the mapping $\mathcal{V}_n = \mathcal{V}_{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}^n$ be defined as in Corollary 3.5. Furthermore, let the operators $(\mathcal{D}^c)^n$ and $(\mathcal{S}^c)^n$ be defined as in* (2.3.13) *and* (2.3.17)*, respectively. Then for $\Psi \in \mathcal{M}$, $\Phi \in \mathcal{M}^{n+1}$ we get*

$$\mathcal{V}_n\Psi = O_{g_n}(\mathcal{D}^c)^n \tag{3.1.14}$$

with $g_n : M^{n+1} \longrightarrow \mathbb{C}$,

$$g_n(\varphi_0, \ldots, \varphi_n) := \prod_{k=0}^{n-1} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n), \tag{3.1.15}$$

and

$$(\mathcal{V}_n)^* \Phi = (\mathcal{S}^c)^n O_{\overline{g_n}}. \tag{3.1.16}$$

PROOF. We will prove the first part by induction.
Let $n = 2$. According to Corollary 3.5 we have $\mathcal{V}_2 = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V})\mathcal{V}$. Hence for $\Psi \in \mathcal{M}$, φ_0, φ_1, $\varphi_2 \in M$

$$\begin{aligned}
&\mathcal{V}_2\Psi(\varphi_0, \varphi_1, \varphi_2) = [(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V})\mathcal{V}\Psi](\varphi_0, \varphi_1, \varphi_2) = \mathcal{V}(\mathcal{V}\Psi(\varphi_0, \cdot))(\varphi_1, \varphi_2) \\
&\quad = g(\varphi_1, \varphi_2) \cdot \mathcal{V}\Psi(\varphi_0, \varphi_1 + \varphi_2) = g(\varphi_1, \varphi_2) \cdot g(\varphi_0, \varphi_1 + \varphi_2) \cdot \Psi(\varphi_0 + \varphi_1 + \varphi_2) \\
&\quad = g_2(\varphi_0, \varphi_1, \varphi_2) \cdot \Psi(\varphi_0 + \varphi_1 + \varphi_2).
\end{aligned}$$

Now let $n \in \mathbb{N}$ and assume $g_n(\varphi_0, \ldots, \varphi_n) = \prod_{k=0}^{n-1} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n)$ for all $\varphi_0, \ldots, \varphi_n \in M$. From Corollary 3.5 we know $\mathcal{V}_{n+1} = (\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_n)\mathcal{V}$.
Let $\varphi_0, \ldots, \varphi_{n+1} \in M$ and $\Psi \in \mathcal{M}$.

$$\begin{aligned}
&\mathcal{V}_{n+1}\Psi \quad (\varphi_0, \ldots, \varphi_{n+1}) \\
&\quad = [(\mathbb{1}_{\mathcal{B}} \otimes \mathcal{V}_n)\mathcal{V}\Psi](\varphi_0, \ldots, \varphi_{n+1}) = \mathcal{V}_n(\mathcal{V}\Psi(\varphi_0, \cdot))(\varphi_1, \ldots, \varphi_{n+1}) \\
&\quad = g_n(\varphi_1, \ldots, \varphi_{n+1}) \cdot \mathcal{V}\Psi(\varphi_0, \varphi_1 + \ldots + \varphi_{n+1}) \\
&\quad = g_n(\varphi_1, \ldots, \varphi_{n+1}) \cdot g(\varphi_0, \varphi_1 + \ldots + \varphi_{n+1}) \cdot \Psi(\varphi_0 + \ldots + \varphi_{n+1}) \\
&\quad = \prod_{k=1}^{n} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_{n+1}) \cdot g(\varphi_0, \varphi_1 + \ldots + \varphi_{n+1}) \cdot \Psi(\varphi_0 + \ldots + \varphi_{n+1}) \\
&\quad = \prod_{k=0}^{n} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n) \cdot \Psi(\varphi_0 + \ldots + \varphi_{n+1}) \\
&\quad = g_{n+1}(\varphi_0, \ldots, \varphi_{n+1})\Psi(\varphi_0 + \ldots + \varphi_{n+1}).
\end{aligned}$$

So, the formula holds for all $n \in \mathbb{N}$.
Now let $n \in \mathbb{N}$, $\Psi \in \mathcal{M}$, $\Phi \in \mathcal{M}^{n+1}$. Because of Corollary 3.5 and the first part of the proof we get using Proposition 2.11 twice

$$\begin{aligned}
&\langle \Psi, \mathcal{V}_n^* \Phi \rangle_{\mathcal{M}} = \langle \mathcal{V}_n \Psi, \Phi \rangle_{\mathcal{M}^{n+1}} \\
&= \int F^{n+1}(d[\varphi_0, \ldots, \varphi_n]) \overline{\mathcal{V}_n \Psi(\varphi_0, \ldots, \varphi_n)} \Phi(\varphi_0, \ldots, \varphi_n) \\
&= \int F^{n+1}(d[\varphi_0, \ldots, \varphi_n]) \overline{g_n(\varphi_0, \ldots, \varphi_n)\Psi(\varphi_0 + \ldots + \varphi_n)} \cdot \Phi(\varphi_0, \ldots, \varphi_n) \\
&= \int F(d\varphi_n) \sum_{\varphi_0 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_1}{\varphi_0} \cdots \binom{\varphi_n}{\varphi_{n-1}} \cdot \overline{g_n(\varphi_0, \varphi_1 - \varphi_0, \ldots, \varphi_n - \varphi_{n-1})} \cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot \overline{\Psi}(\varphi_n)\Phi(\varphi_0, \varphi_1 - \varphi_0, \ldots, \varphi_n - \varphi_{n-1}) \\
= &\int F(d\varphi)\ \overline{\Psi}(\varphi) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n \subseteq \varphi} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_n}{\varphi_{n-1}} \binom{\varphi}{\varphi_n} \cdot \\
&\cdot \overline{g_n(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}, \varphi - \varphi_n)} \cdot \Phi(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}, \varphi - \varphi_n).
\end{aligned}
$$

So, $\mathcal{V}_n^*\Phi$ has the above form. □

Now we return to generalized splitting procedures described by a transition expectation $\mathcal{E}_{U_1,U_2}$ with isometric operators U_1, U_2 from $\mathfrak{L}(\mathcal{M})$.

Proposition 3.7. *Let $n \in \mathbb{N}$ and U_1, U_2 isometric operators on $\mathcal{M}$. Furthermore, let $\mathcal{V}^n_{U_1,U_2}$ defined as in Proposition 3.4, the function g_n be defined by* (3.1.15)*. Then*

$$\mathcal{V}^n_{U_1,U_2} = (U_1^{\otimes n} \otimes U_2)\mathcal{V}_n = (U_1^{\otimes n} \otimes U_2)O_{g_n}(\mathcal{D}^c)^n \tag{3.1.17}$$

where $U_1^{\otimes n}$ is the operator on $\mathcal{M}^n$ given by

$$U_1^{\otimes n}(\Psi_1 \otimes \ldots \otimes \Psi_n) := U_1\Psi_1 \otimes \ldots \otimes U_1\Psi_n \qquad (\Psi_i \in \mathcal{M}\ \forall i \in n]). \tag{3.1.18}$$

The adjoint operator $(\mathcal{V}^n_{U_1,U_2})^*$ is given by

$$(\mathcal{V}^n_{U_1,U_2})^* = \mathcal{V}_n^*((U_1^*)^{\otimes n} \otimes U_2^*) = (\mathcal{S}^c)^n O_{\overline{g_n}}((U_1^*)^{\otimes n} \otimes U_2^*). \tag{3.1.19}$$

PROOF. We prove (3.1.17) by induction. For $n = 1$ we get from (3.1.10) and Proposition 3.6

$$\mathcal{V}^1_{U_1,U_2} = (U_1 \otimes U_2)\mathcal{V} = (U_1 \otimes U_2)O_g\mathcal{D}^c. \tag{3.1.20}$$

Assume (3.1.17) holds for $n \in \mathbb{N}$. Using (3.1.9) and (3.1.14) we get

$$
\begin{aligned}
\mathcal{V}^{n+1}_{U_1,U_2} &= (\mathbb{1} \otimes \mathcal{V}^n_{U_1,U_2})\mathcal{V}^1_{U_1,U_2} = (\mathbb{1} \otimes (U_1^{\otimes n} \otimes U_2)\mathcal{V}_n)\mathcal{V}^1_{U_1,U_2} \\
&= (\mathbb{1} \otimes (U_1^{\otimes n} \otimes U_2)\mathcal{V}_n)(U_1 \otimes U_2)\mathcal{V} \\
&= (U_1 \otimes (U_1^{\otimes n} \otimes U_2)\mathcal{V}_n)\mathcal{V} = (U_1^{\otimes(n+1)} \otimes U_2)(\mathbb{1}^{\otimes(n+1)} \otimes \mathcal{V}_n)\mathcal{V} \\
&= (U_1^{\otimes(n+1)} \otimes U_2)\mathcal{V}_{n+1} = (U_1^{\otimes(n+1)} \otimes U_2)O_{g_{n+1}}(\mathcal{D}^c)^{n+1}.
\end{aligned}
$$

$(\mathcal{V}^n_{U_1,U_2})^*$ is calculated for $n \in \mathbb{N}$ using (3.1.17) and (6.1.11):

$$(\mathcal{V}^n_{U_1,U_2})^* = [(U_1^{\otimes n} \otimes U_2)\mathcal{V}_n]^* = \mathcal{V}_n^*((U_1^*)^{\otimes n} \otimes U_2^*) = (\mathcal{S}^c)^n O_{\overline{g_n}}((U_1^*)^{\otimes n} \otimes U_2^*).$$

□

Example 3.1. *Let $U_1 := O_{h_1}$ and $U_2 := O_{h_2}$ with functions h_1, $h_2 : M \longrightarrow \mathbb{C}$, $|h_1(\varphi)| = 1$ and $|h_2(\varphi)| = 1$ for all $\varphi \in M$ (for instance $h_j(\varphi) = e^{i \cdot r_j(\varphi)}$ with functions $r_j : M \longrightarrow \mathbb{R}$, $j = 1, 2$). Then U_1 and U_2 are isometries and for all $n \in \mathbb{N}$ and $\Psi \in \mathcal{M}$, $\underline{\varphi} := (\varphi_0, \ldots, \varphi_n) \in M^{n+1}$*

$$\mathcal{V}^n_{O_{h_1},O_{h_2}}\Psi(\underline{\varphi}) = \left(\prod_{k=0}^{n-1} h_1(\varphi_k)\right) \cdot h_2(\varphi_n) \cdot g_n(\underline{\varphi})\Psi(\varphi_0 + \ldots + \varphi_n). \tag{3.1.21}$$

PROOF. According to the definition we have

$$(U_j\Psi)(\varphi) = h_j(\varphi)\cdot\Psi(\varphi) \qquad (\Psi\in\mathcal{M},\ \varphi\in M,\ j=1,2).$$

We get for all Ψ, $\Phi\in\mathcal{M}$, $j=1,2$

$$\begin{aligned}\langle U_j\Psi, U_j\Phi\rangle_{\mathcal{M}} &= \int F(d\varphi)\overline{h_j(\varphi)\Psi(\varphi)}h_j(\varphi)\Phi(\varphi) = \int F(d\varphi)|h_j(\varphi)|^2\overline{\Psi(\varphi)}\Phi(\varphi)\\ &= \int F(d\varphi)\overline{\Psi(\varphi)}\Phi(\varphi) = \langle\Psi,\Phi\rangle_{\mathcal{M}}.\end{aligned}$$

hence, U_1 and U_2 are isometries.
Furthermore, we get for $\Psi\in\mathcal{M}$, $\underline{\varphi} := (\varphi_0,\dots,\varphi_n)\in M^{n+1}$

$$\begin{aligned}\mathcal{V}^n_{U_1,U_2}\Psi(\underline{\varphi}) &= (U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi(\underline{\varphi}) = (O_{h_1}^{\otimes n}\otimes O_{h_2})O_{g_n}(\mathcal{D}^c)^n\Psi(\underline{\varphi})\\ &= \left(\prod_{i=0}^{n-1} h_1(\varphi_k)\right)\cdot h_2(\varphi_n)\cdot g_n(\underline{\varphi})\cdot\Psi(\varphi_0+\ldots+\varphi_n).\end{aligned}$$

$\square$

3.2 Quantum Markov Chains with Transition Expectation $\mathcal{E}_{U_1,U_2}$

Now we want to consider quantum Markov chains for generalized splitting procedures including additional independent evolutions of the quantum system and the measurement apparatus (described by isometric operators U_1 and U_2 on $\mathcal{A}=\mathcal{B}=\mathfrak{L}(\mathcal{M})$). The corresponding transition expectation was defined in (3.1.6).
We will be able to give explicit formulae for kernels of the density matrices of the states of the measurement apparatus up to time n and at time n for $n\in\mathbb{N}$.
According to the definitions in chapter 1 the states of the measurement apparatus up to time n and at time n for $n\in\mathbb{N}$ are described as follows

$$\begin{aligned}\omega_{n]}^{U_1,U_2}(B_1\otimes\ldots\otimes B_n) &= \tau(\mathcal{E}^n_{U_1,U_2}(B_1\otimes\ldots\otimes B_n\otimes\mathbb{1}_{\mathcal{A}}))\\ &= \tau((\mathcal{V}^n_{U_1,U_2})^*(B_1\otimes\ldots\otimes B_n\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}), \qquad (3.2.1)\\ \omega_n^{U_1,U_2}(B) &= \tau(\mathcal{E}^n_{U_1,U_2}(\mathbb{1}_{\mathcal{B}}^{n-1}\otimes B\otimes\mathbb{1}_{\mathcal{A}}))\\ &= \tau((\mathcal{V}^n_{U_1,U_2})^*(\mathbb{1}_{\mathcal{B}}^{n-1}\otimes B\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}) \qquad (3.2.2)\end{aligned}$$

with initial state τ, transition expectation $\mathcal{E}_{U_1,U_2}$ and $B_1,\dots B_n, B\in\mathcal{B}$.
For the proofs in this section we will use the following remark (for details see [37]).

Remark 3.8. *For normal states ω on $\mathfrak{L}(\mathcal{M})$ there exists a sequence $(\Psi_k)_{k\in\mathbb{N}}$ from $\mathcal{M}$ with $\sum_{k\in\mathbb{N}}\|\Psi_k\|^2<\infty$ and $\omega=\sum_{k\in\mathbb{N}}\langle\Psi_k,(.)\Psi_k\rangle$.*
The density matrix K of ω is an operator on $\mathcal{M}$ defined by $K:=\sum_{k\in\mathbb{N}}\langle\Psi_k,(.)\rangle\Psi_k$.

So, for $\Psi \in \mathcal{M}$ and $\varphi \in M$ we have

$$\begin{aligned}(K\Psi)(\varphi) &= \sum_{k\in\mathbb{N}} \langle \Psi_k, \Psi\rangle \Psi_k(\varphi) = \int F(d\widetilde{\varphi}) \sum_{k\in\mathbb{N}} \overline{\Psi_k}(\widetilde{\varphi})\Psi(\widetilde{\varphi})\Psi_k(\varphi) \\ &= \int F(d\widetilde{\varphi})\Psi(\widetilde{\varphi}) \sum_{k\in\mathbb{N}} \overline{\Psi_k}(\widetilde{\varphi})\Psi_k(\varphi) = \int F(d\widetilde{\varphi})\Psi(\widetilde{\varphi})\rho(\varphi,\widetilde{\varphi}) \qquad (3.2.3)\end{aligned}$$

with $\rho : M \times M \longrightarrow \mathbb{C}$, $\rho(\varphi,\widetilde{\varphi}) = \sum_{k\in\mathbb{N}} \overline{\Psi_k}(\widetilde{\varphi})\Psi_k(\varphi)$ being a kernel of K.
For integral operators $A \in \mathfrak{L}(\mathcal{M})$ with kernel $a(.,.)$ we get

$$\begin{aligned}\omega(A) &= \sum_{k\in\mathbb{N}} \langle \Psi_k, A\Psi_k\rangle = \sum_{k\in\mathbb{N}} \int F(d\varphi)\overline{\Psi_k}(\varphi)(A\Psi_k)(\varphi) \\ &= \sum_{k\in\mathbb{N}} \int F(d\varphi)\overline{\Psi_k}(\varphi)\int F(d\widetilde{\varphi})a(\varphi,\widetilde{\varphi})\Psi_k(\widetilde{\varphi}) = \int F(d\varphi)\int F(d\widetilde{\varphi})a(\varphi,\widetilde{\varphi}) \sum_{k\in\mathbb{N}} \overline{\Psi_k}(\varphi)\Psi_k(\widetilde{\varphi}) \\ &= \int F(d\varphi) \int F(d\widetilde{\varphi})a(\varphi,\widetilde{\varphi})\rho(\widetilde{\varphi},\varphi). \qquad (3.2.4)\end{aligned}$$

Proposition 3.9. *If τ is a normal state on $\mathcal{A}$ then for all $n \geq 1$ $\omega_{n]}^{U_1,U_2}$ is a normal state on $\mathcal{B}^n$.*

PROOF. Because τ is a normal state according to Remark 3.8 there exists a sequence $(\Psi_k)_{k\in N}$ from $\mathcal{M}$ with $\sum_{k\in N} \|\Psi_k\|^2 < \infty$, such that for $A \in \mathcal{A}$ there holds

$$\tau(A) = \sum_{k\in\mathbb{N}} \langle \Psi_k, A\Psi_k\rangle_{\mathcal{M}}. \qquad (3.2.5)$$

So, for all $B_1, \ldots, B_n \in \mathcal{B}$ we have

$$\begin{aligned}\omega_{n]}^{U_1,U_2}(B_1 \otimes \ldots \otimes B_n) &= \tau((\mathcal{V}^n_{U_1,U_2})^*(B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}) \\ &= \sum_{k\in\mathbb{N}} \langle \Psi_k \ , \ (\mathcal{V}^n_{U_1,U_2})^*(B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k\rangle_{\mathcal{M}} \\ &= \sum_{k\in\mathbb{N}} \langle \mathcal{V}^n_{U_1,U_2}\Psi_k \ , \ (B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k\rangle_{\mathcal{M}^{n+1}}. \qquad (3.2.6)\end{aligned}$$

$\mathcal{V}^n_{U_1,U_2} : \mathcal{M} \longrightarrow \mathcal{M}^{n+1}$ is an isometry (see Proposition 3.4). This implies

$$\sum_{k\in N} \|\mathcal{V}^n_{U_1,U_2}\Psi_k\|^2 \ = \ \sum_{k\in N} \|\Psi_k\|^2 < \infty \ .$$

Hence, there exists a sequence $(\mathcal{V}^n_{U_1,U_2}\Psi_k)_{k\in N}$ from $\mathcal{M}^{n+1}$ with $\sum_{k\in N} \|\mathcal{V}^n_{U_1,U_2}\Psi_k\|^2 < \infty$ and

$$\begin{aligned}&\tau((\mathcal{V}^n_{U_1,U_2})^*(B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}) \\ &\qquad = \sum_{k\in\mathbb{N}} \langle \mathcal{V}^n_{U_1,U_2}\Psi_k, (B_1 \otimes \ldots \otimes B_n \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k\rangle_{\mathcal{M}^{n+1}}. \qquad (3.2.7)\end{aligned}$$

$\omega^{U_1,U_2}_{n]} = \tau(\mathcal{E}^n_{U_1,U_2}(\cdot \otimes \mathbb{1}_{\mathcal{A}}))$ is therefore σ-weak continuous and hence a normal state on $\mathcal{B}^n$. □

(3.2.6) allows to give explicitly the density matrices of the normal states $\omega^{U_1,U_2}_{n]}$ and ω^{U_1,U_2}_{n}:
Because of (3.2.7), $\omega^{U_1,U_2}_{n]} = \tau(\mathcal{E}^n_{U_1,U_2}(\cdot \otimes \mathbb{1}_{\mathcal{A}}))$ is a normal state on $\mathcal{B}^n$. Hence, there exists a density matrix $K^{U_1,U_2}_{n]}$ on $\mathcal{M}^n$ with $\omega^{U_1,U_2}_{n]}(\cdot) = Tr(K^{U_1,U_2}_{n]}(\cdot)\ )$.
$K^{U_1,U_2}_{n]}$ is an operator on $\mathcal{M}^n$ with

$$K^{U_1,U_2}_{n]}\Psi(\underline{\varphi}) = \int F^n(d\underline{\widetilde{\varphi}})\Psi(\underline{\widetilde{\varphi}})\rho^{U_1,U_2}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}}), \tag{3.2.8}$$

where $\rho^{U_1,U_2}_{n]} : M^n \times M^n \longrightarrow \mathbb{C}$ is a kernel of $K^{U_1,U_2}_{n]}$.

Proposition 3.10. *Let $n \in \mathbb{N}$, $\underline{\varphi} := (\varphi_1, \dots, \varphi_n)$, $\underline{\widetilde{\varphi}} := (\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \in M^n$ and let U_1, U_2 be isometric operators on $\mathcal{M}$. Furthermore, let $\rho^{U_1,U_2}_{n]}$ be a kernel defined in* (3.2.8)*. Then for an initial state τ satisfying* (3.2.5) *it follows*

$$\begin{aligned}&\rho^{U_1,U_2}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}})\\ =&\int F(d\varphi)\sum_{k\in\mathbb{N}} \overline{(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\widetilde{\varphi}},\varphi)}(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\varphi},\varphi).\end{aligned} \tag{3.2.9}$$

PROOF. Using the notations from the proof of Proposition 3.9 we get

$$\begin{aligned}\omega^{U_1,U_2}_{n]}(\cdot) &= \sum_{k\in\mathbb{N}}\langle \mathcal{V}^n_{U_1,U_2}\Psi_k, (\cdot\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k\rangle_{\mathcal{M}^{n+1}}\\
&= \sum_{k\in\mathbb{N}}\int F(d\varphi)\int F^n(d\underline{\varphi})\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k}(\underline{\varphi},\varphi)(\cdot\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)\\
&= \int F(d\varphi)\sum_{k\in\mathbb{N}}\int F^n(d\underline{\varphi})\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k}(\underline{\varphi},\varphi)(\cdot\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)\\
&= \int F(d\varphi)\sum_{k\in\mathbb{N}}\int F^n(d\underline{\varphi})\overline{(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k}(\underline{\varphi},\varphi)\\
&\qquad (.\otimes\mathbb{1}_{\mathcal{A}})(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\varphi},\varphi).\end{aligned} \tag{3.2.10}$$

The density matrix $K^{U_1,U_2}_{n]}$ of $\omega^{U_1,U_2}_{n]}$ is an operator on $\mathcal{M}^n$ with

$$\begin{aligned}K^{U_1,U_2}_{n]}\Psi(\underline{\varphi}) &= \int F(d\varphi)\sum_{k\in\mathbb{N}}\langle\mathcal{V}^n_{U_1,U_2}\Psi_k(\cdot,\varphi),\Psi\rangle_{\mathcal{M}^n}\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)\\
&= \int F(d\varphi)\sum_{k\in\mathbb{N}}\int F^n(d\underline{\widetilde{\varphi}})\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi)}\Psi(\underline{\widetilde{\varphi}})\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)\\
&= \int F(d\varphi)\int F^n(d\underline{\widetilde{\varphi}})\Psi(\underline{\widetilde{\varphi}})\sum_{k\in\mathbb{N}}\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi)}\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)\\
&= \int F^n(d\underline{\widetilde{\varphi}})\Psi(\underline{\widetilde{\varphi}})\rho^{U_1,U_2}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}}),\end{aligned}$$

where

$$\rho^{U_1,U_2}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}}) = \int F(d\varphi)\sum_{k\in\mathbb{N}} \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi)}\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\varphi},\varphi)$$

$$=\int F(d\varphi)\sum_{k\in\mathbb{N}} \overline{(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\widetilde{\varphi}},\varphi)}(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\varphi},\varphi).$$

□

Remark 3.11. *For integral operators* $\underline{A}$ *on* $\mathcal{M}^n$ *with kernel* $\underline{a}(.,.)$ (3.2.10) *becomes*

$$\omega^{U_1,U_2}_{n]}(\underline{A}) = \int F(d\varphi)\sum_{k\in\mathbb{N}}\int F^n(d\underline{\varphi})\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k}(\underline{\varphi},\varphi)\int F(d\underline{\widetilde{\varphi}})\underline{a}(\underline{\varphi},\underline{\widetilde{\varphi}})\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi)$$

$$= \int F(d\varphi)\int F^n(d\underline{\varphi})\int F^n(d\underline{\widetilde{\varphi}})\underline{a}(\underline{\varphi},\underline{\widetilde{\varphi}})\sum_{k\in\mathbb{N}}\overline{\mathcal{V}^n_{U_1,U_2}\Psi_k}(\underline{\varphi},\varphi)\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi)$$

$$= \int F^n(d\underline{\varphi})\int F^n(d\underline{\widetilde{\varphi}})\underline{a}(\underline{\varphi},\underline{\widetilde{\varphi}})\rho^{U_1,U_2}_{n]}(\underline{\widetilde{\varphi}},\underline{\varphi}). \tag{3.2.11}$$

Example 3.2. *Let* $n\in\mathbb{N}$ *and* $U_1 := O_{h_1}$, $U_2 := O_{h_2}$ *with functions* $h_j : M \longrightarrow \mathbb{C}$, $|h_j(\varphi)| = 1$ *for all* $\varphi\in M$, $j = 1,2$.
Then for $\underline{\varphi} := (\varphi_1,\ldots,\varphi_n)$, $\underline{\widetilde{\varphi}} := (\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_n)\in M^n$ *there holds*

$$\rho^{O_{h_1},O_{h_2}}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}}) \tag{3.2.12}$$

$$= \int F(d\varphi)\overline{g_n}(\underline{\widetilde{\varphi}},\varphi)g_n(\underline{\varphi},\varphi)\prod_{i=1}^n\Big(\overline{h_1(\widetilde{\varphi}_i)}h_1(\varphi_i)\Big)\cdot\rho\Big(\sum_{i=1}^n\widetilde{\varphi}_i+\varphi,\sum_{i=1}^n\varphi_i+\varphi\Big),$$

where ρ *is defined by* (3.2.3).

PROOF. We insert $U_1 = O_{h_1}$ and $U_2 = O_{h_2}$ into (3.2.9) and get

$$\rho^{U_1,U_2}_{n]}(\underline{\varphi},\underline{\widetilde{\varphi}})$$

$$= \int F(d\varphi)\sum_{k\in\mathbb{N}}\overline{(O_{h_1}^{\otimes n}\otimes O_{h_2})O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\widetilde{\varphi}},\varphi)}(O_{h_1}^{\otimes n}\otimes O_{h_2})O_{g_n}(\mathcal{D}^c)^n\Psi_k(\underline{\varphi},\varphi)$$

$$= \int F(d\varphi)\sum_{k\in\mathbb{N}}\Bigg[\overline{\Big(\prod_{i=1}^n h_1(\widetilde{\varphi}_i)\Big)h_2(\varphi)g_n(\underline{\widetilde{\varphi}},\varphi)\Psi_k\Big(\sum_{i=1}^n\widetilde{\varphi}_i+\varphi\Big)}\cdot$$

$$\cdot\Big(\prod_{i=1}^n h_1(\varphi_i)\Big)h_2(\varphi)g_n(\underline{\varphi},\varphi)\Psi_k\Big(\sum_{i=1}^n\varphi_i+\varphi\Big)\Bigg]$$

$$=\int F(d\varphi)\overline{g_n(\underline{\widetilde{\varphi}},\varphi)}g_n(\underline{\varphi},\varphi)\Big(\prod_{i=1}^n\overline{h_1(\widetilde{\varphi}_i)}h_1(\varphi_i)\Big)\sum_{k\in\mathbb{N}}\Big[\overline{\Psi_k\Big(\sum_{i=1}^n\widetilde{\varphi}_i+\varphi\Big)}\Psi_k\Big(\sum_{i=1}^n\varphi_i+\varphi\Big)\Big]$$

$$=\int F(d\varphi)\overline{g_n(\underline{\widetilde{\varphi}},\varphi)}g_n(\underline{\varphi},\varphi)\Big(\prod_{i=1}^n\overline{h_1(\widetilde{\varphi}_i)}h_1(\varphi_i)\Big)\cdot\rho\Big(\sum_{i=1}^n\widetilde{\varphi}_i+\varphi,\sum_{i=1}^n\varphi_i+\varphi\Big).$$

□

Proposition 3.12. *Let $g : M \times M \longrightarrow \mathbb{C}$ be a function satisfying the isometry condition* (3.1.1)*. Furthermore, let ρ be given by* (3.2.3) *and $\rho_{n]}$ be defined as in* (3.2.9) *with $U_1 = \mathbb{1}_{\mathcal{B}}$ and $U_2 = \mathbb{1}_{\mathcal{A}}$. Then for all $\underline{\varphi}^0 = (\varphi_1^0, \dots, \varphi_n^0)$ and $\underline{\varphi}^1 = (\varphi_1^1, \dots, \varphi_n^1)$ from M^n there holds*

$$\rho_{n]}(\underline{\varphi}^0, \underline{\varphi}^1) \quad = \quad \int F(d\varphi)\overline{g_n(\underline{\varphi}^1, \varphi)}g_n(\underline{\varphi}^0, \varphi) \cdot \rho(\varphi + \sum_{l=1}^{n} \varphi_l^0, \varphi + \sum_{l=1}^{n} \varphi_l^1).$$

with g_n defined by (3.1.15)*.*

PROOF. According to (3.2.3) we may assume that ρ is of the form

$$\rho(\varphi_0, \varphi_1) = \sum_{k \in \mathbb{N}} \Psi_k(\varphi_0)\overline{\Psi_k(\varphi_1)}. \tag{3.2.13}$$

with a sequence $(\Psi_k)_{k \in \mathbb{N}}$ from $\mathcal{M}$ satisfying $\sum\limits_{k \in \mathbb{N}} \|\Psi_k\|^2 < \infty$.
From (3.2.9) with $U_1 = \mathbb{1}_{\mathcal{B}}$ and $U_2 = \mathbb{1}_{\mathcal{A}}$ and Proposition 3.6 we get for $\underline{\varphi}^0 = (\varphi_1^0, \dots, \varphi_n^0)$ and $\underline{\varphi}^1 = (\varphi_1^1, \dots, \varphi_n^1)$ from M^n

$$\begin{aligned}
\rho_{n]}(\underline{\varphi}^0, \underline{\varphi}^1) &= \int F(d\varphi) \sum_{k \in \mathbb{N}} \overline{\mathcal{V}_n \Psi_k(\underline{\varphi}^1, \varphi)} \mathcal{V}_n \Psi_k(\underline{\varphi}^0, \varphi) \\
&= \int F(d\varphi)\overline{g_n(\underline{\varphi}^1, \varphi)}g_n(\underline{\varphi}^0, \varphi) \cdot \sum_{r \in \mathbb{N}} \Psi_r\left(\varphi + \varphi_1^0 + \dots + \varphi_n^0\right) \cdot \overline{\Psi_r\left(\varphi + \varphi_1^1 + \dots + \varphi_n^1\right)} \\
&= \int F(d\varphi)\overline{g_n(\underline{\varphi}^1, \varphi)}g_n(\underline{\varphi}^0, \varphi) \cdot \rho(\varphi + \sum_{l=1}^{n} \varphi_l^0, \varphi + \sum_{l=1}^{n} \varphi_l^1).
\end{aligned}$$

where we used the definition (3.1.15) of the function g_n. □

Proposition 3.13. *Let ρ be a kernel of the initial state τ having the form* (3.2.13)*. For all $n \geq 1$ and isometric operators U_1, U_2 on $\mathcal{M}$ let $\rho_n^{U_1,U_2}$ be a kernel of the density matrix of the normal state $\omega_n^{U_1,U_2}$ on $\mathcal{B}$ with*

$$\omega_n^{U_1,U_2}(A) = \tau(\mathcal{E}_{U_1,U_2}^n(\mathbb{1}_{\mathcal{B}}^{n-1} \otimes A \otimes \mathbb{1}_{\mathcal{A}})) = \tau((\mathcal{V}_{U_1,U_2}^n)^*(\mathbb{1}_{\mathcal{B}}^{n-1} \otimes A \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}_{U_1,U_2}^n) \quad (A \in \mathcal{B}). \tag{3.2.14}$$

Then for all $\varphi_0, \hat{\varphi}_0 \in M$

$$\begin{aligned}
\rho_n^{U_1,U_2}(\varphi_0, \hat{\varphi}_0) = \int F^n(d\underline{\varphi}) \sum_{k \in \mathbb{N}} & \overline{(U_1^{\otimes n} \otimes U_2) O_{g_n}(\mathcal{D}^c)^n \Psi_k(\varphi_1, \dots, \varphi_{n-1}, \widehat{\varphi}_0, \varphi_n)} \\
& (U_1^{\otimes n} \otimes U_2) O_{g_n}(\mathcal{D}^c)^n \Psi_k(\varphi_1, \dots, \varphi_{n-1}, \varphi_0, \varphi_n)
\end{aligned} \tag{3.2.15}$$

where for $n \in \mathbb{N}$ we used again the notations $\underline{\varphi} := (\varphi_1, \dots, \varphi_n)$ and (3.1.15)*.*

PROOF. Let A be an integral operator from $\mathcal{B}$ with kernel $a : M \times M \longrightarrow \mathbb{C}$. Furthermore, let for $n \in \mathbb{N}$ $\underline{\widetilde{\varphi}} := (\varphi_1, \ldots, \varphi_{n-1})$ be a vector from M^{n-1}. For all $\Phi, \Psi \in \mathcal{M}$ we get

$$\begin{aligned}
&\langle \Phi, (\mathcal{V}^n_{U_1,U_2})^* (\mathbb{1}^{n-1}_{\mathcal{B}} \otimes A \otimes \mathbb{1}_{\mathcal{A}}) \mathcal{V}^n_{U_1,U_2} \Psi \rangle_{\mathcal{M}} = \langle \mathcal{V}^n_{U_1,U_2} \Phi, (\mathbb{1}^{n-1}_{\mathcal{B}} \otimes A \otimes \mathbb{1}_{\mathcal{A}}) \mathcal{V}^n_{U_1,U_2} \Psi \rangle_{\mathcal{M}^{n+1}} \\
&= \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\varphi_1) \int F(d\hat{\varphi}) \overline{\mathcal{V}^n_{U_1,U_2}\Phi}(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi}) (\mathbb{1}^{n-1}_{\mathcal{B}} \otimes A \otimes \mathbb{1}_{\mathcal{A}}) \mathcal{V}^n_{U_1,U_2} \Psi(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi}) \\
&= \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\varphi_1) \int F(d\hat{\varphi}) \overline{\mathcal{V}^n_{U_1,U_2}\Phi(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} A (\mathcal{V}^n_{U_1,U_2} \Psi(\underline{\widetilde{\varphi}}, \cdot, \hat{\varphi}))(\varphi_1) \\
&= \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\varphi_1) \int F(d\hat{\varphi}) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \overline{\mathcal{V}^n_{U_1,U_2}\Phi(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} \mathcal{V}^n_{U_1,U_2} \Psi(\underline{\widetilde{\varphi}}, \varphi_0, \hat{\varphi}) \\
&= \int F(d\varphi_1) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\hat{\varphi}) \overline{\mathcal{V}^n_{U_1,U_2}\Phi(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} \mathcal{V}^n_{U_1,U_2} \Psi(\underline{\widetilde{\varphi}}, \varphi_0, \hat{\varphi}).
\end{aligned} \tag{3.2.16}$$

Let

$$\rho(\varphi_0, \varphi_1) = \sum_{k \in \mathbb{N}} \Psi_k(\varphi_0) \overline{\Psi_k(\varphi_1)} \tag{3.2.17}$$

be a kernel of the density matrix of τ with an at most countable sequence $(\Psi_k)_{k\in\mathbb{N}}$ from $\mathcal{M}$ with $\sum\limits_{k\in\mathbb{N}} \|\Psi_k\|^2 < \infty$. Then we get from (3.2.16) and (3.2.18)

$$\begin{aligned}
\omega^{U_1,U_2}_n(A) &= \sum_{k\in\mathbb{N}} \langle \Psi_k, (\mathcal{V}^n_{U_1,U_2})^* (\mathbb{1}^{n-1}_{\mathcal{B}} \otimes A \otimes \mathbb{1}_{\mathcal{A}}) \mathcal{V}^n_{U_1,U_2} \Psi_k \rangle_{\mathcal{M}} \\
&= \sum_{k\in\mathbb{N}} \int F(d\varphi_1) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \\
&\qquad \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\hat{\varphi}) \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} \mathcal{V}^n_{U_1,U_2} \Psi_k(\underline{\widetilde{\varphi}}, \varphi_0, \hat{\varphi}) \\
&= \int F(d\varphi_1) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \\
&\qquad \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\hat{\varphi}) \sum_{k\in\mathbb{N}} \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} \mathcal{V}^n_{U_1,U_2} \Psi_k(\underline{\widetilde{\varphi}}, \varphi_0, \hat{\varphi}) \\
&= \int F(d\varphi_1) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \rho^{U_1,U_2}_n(\varphi_0, \varphi_1)
\end{aligned}$$

with

$$\rho^{U_1,U_2}_n(\varphi_0, \varphi_1) := \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\hat{\varphi}) \sum_{k\in\mathbb{N}} \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}}, \varphi_1, \hat{\varphi})} \mathcal{V}^n_{U_1,U_2} \Psi_k(\underline{\widetilde{\varphi}}, \varphi_0, \hat{\varphi}).$$

Hence we have

$$\omega^{U_1,U_2}_n(A) = \int F(d\varphi_1) \int F(d\varphi_0) a(\varphi_1, \varphi_0) \rho^{U_1,U_2}_n(\varphi_0, \varphi_1)$$

with a kernel $\rho_n^{U_1,U_2}$ given by

$$\rho_n^{U_1,U_2}(\varphi_0,\hat{\varphi}_0) = \int F^{n-1}(d\underline{\widetilde{\varphi}}) \int F(d\varphi_n) \sum_{k\in\mathbb{N}} \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\hat{\varphi}_0,\varphi_n)} \cdot \mathcal{V}^n_{U_1,U_2}\Psi_k(\underline{\widetilde{\varphi}},\varphi_0,\varphi_n)$$
$$= \int F^n(d\underline{\varphi}) \sum_{k\in\mathbb{N}} \overline{\mathcal{V}^n_{U_1,U_2}\Psi_k(\varphi_1,\ldots\varphi_{n-1},\hat{\varphi}_0,\varphi_n)} \mathcal{V}^n_{U_1,U_2}\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n).$$

From (3.1.17) we know that

$$\mathcal{V}^n_{U_1,U_2}\Psi_k = (U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k.$$

With (3.2.17) we have

$$\rho_n^{U_1,U_2}(\varphi_0,\hat{\varphi}_0) = \int F^n(d\underline{\varphi}) \sum_{k\in\mathbb{N}} \overline{(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\hat{\varphi}_0,\varphi_n)}$$
$$(U_1^{\otimes n}\otimes U_2)O_{g_n}(\mathcal{D}^c)^n\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n).$$

□

Example 3.3. *Let $n\in\mathbb{N}$ and $U_1 := O_{h_1}$, $U_2 := O_{h_2}$ with functions $h_j : M \longrightarrow \mathbb{C}$, $|h_j(\varphi)| = 1$ for all $\varphi\in M$, $j = 1,2$. Furthermore, let φ_0, $\widehat{\varphi}_0 \in M$.*
Then

$$\rho_n^{O_{h_1},O_{h_2}}(\varphi_0,\widehat{\varphi}_0) = \int F^n(d\underline{\varphi})\overline{g_n}(\varphi_1,\ldots,\varphi_{n-1},\widehat{\varphi}_0,\varphi_n) g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\cdot$$
$$\cdot\ \overline{h_1(\widehat{\varphi}_0)}h_1(\varphi_0)\cdot\rho\Bigg(\sum_{i=1}^n\varphi_i+\varphi_0,\sum_{i=1}^n\varphi_i+\widehat{\varphi}_0\Bigg),$$

where $\underline{\varphi} := (\varphi_1,\ldots,\varphi_n)\in M^n$ and ρ is defined by (3.2.3).

PROOF. We insert $U_1 = O_{h_1}$ and $U_2 = O_{h_2}$ into (3.2.15) and get

$$\rho_n^{U_1,U_2}(\varphi_0,\hat{\varphi}_0) = \int F^n(d\underline{\varphi}) \sum_{k\in\mathbb{N}} \Bigg[\overline{(O_{h_1}^{\otimes n}\otimes O_{h_2})O_{g_n}(\mathcal{D}^c)^n\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\widehat{\varphi}_0,\varphi_n)}$$
$$(O_{h_1}^{\otimes n}\otimes O_{h_2})O_{g_n}(\mathcal{D}^c)^n\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\Bigg]$$
$$= \int F^n(d\underline{\varphi}) \sum_{k\in\mathbb{N}} \Bigg[\overline{h_1(\widehat{\varphi}_0)h_2(\varphi_n)\Bigg(\prod_{i=1}^{n-1}h_1(\varphi_i)\Bigg)g_n(\varphi_1,\ldots,\varphi_{n-1},\widehat{\varphi}_0,\varphi_n)\Psi_k\Bigg(\widehat{\varphi}_0+\sum_{i\in\mathbb{N}}\varphi_i\Bigg)}$$
$$h_1(\varphi_0)h_2(\varphi_n)\Bigg(\prod_{i=1}^{n-1}h_1(\varphi_i)\Bigg)g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\Psi_k\Bigg(\varphi_0+\sum_{i\in\mathbb{N}}\varphi_i\Bigg)\Bigg]$$
$$= \int F^n(d\underline{\varphi})\overline{g_n(\varphi_1,\ldots,\varphi_{n-1},\widehat{\varphi}_0,\varphi_n)} g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\overline{(h_1(\widehat{\varphi}_0)}h_1(\varphi_0)\cdot$$

$$\cdot\Big(\prod_{i=1}^{n-1}|h_1(\varphi_i)|^2\Big)\sum_{k\in\mathbb{N}}\overline{\Psi_k\Big(\widehat{\varphi}_0+\sum_{i\in\mathbb{N}}\varphi_i\Big)}\Psi_k\Big(\varphi_0+\sum_{i\in\mathbb{N}}\varphi_i\Big)$$

$$= \int F^n(d\underline{\varphi})\overline{g_n}(\varphi_1,\ldots,\varphi_{n-1},\widehat{\varphi}_0,\varphi_n)g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\cdot\overline{h_1(\widehat{\varphi}_0)}h_1(\varphi_0)\cdot$$

$$\cdot\,\rho\Big(\sum_{i=1}^{n}\varphi_i+\varphi_0,\sum_{i=1}^{n}\varphi_i+\widehat{\varphi}_0\Big).$$

$\square$

Now we calculate the state of the measurement apparatus at time n for the case $U_1=\mathbb{1}_{\mathcal{B}}$, $U_2=\mathbb{1}_{\mathcal{A}}$.

Proposition 3.14. *Let ρ be a kernel of the initial state τ satisfying* (3.2.13). *For all $n\geq 1$ let ρ_n be a kernel of the density matrix of the normal state ω_n on $\mathcal{B}$ with*

$$\omega_n(A)=\tau(\mathcal{E}^n(\mathbb{1}_{\mathcal{B}}^{n-1}\otimes A\otimes\mathbb{1}_{\mathcal{A}}))=\tau(\mathcal{V}_n^*(\mathbb{1}_{\mathcal{B}}^{n-1}\otimes A\otimes\mathbb{1}_{\mathcal{A}})\mathcal{V}_n)\quad(A\in\mathcal{B}).\qquad(3.2.18)$$

Then for all $\varphi_0,\hat{\varphi}_0\in M$

$$\rho_n(\varphi_0,\hat{\varphi}_0) = \int F^n(d\underline{\varphi})\overline{g_n(\varphi_1,\ldots,\varphi_{n-1},\hat{\varphi}_0,\varphi_n)}\cdot g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)$$
$$\cdot\ \rho(\varphi_0+\varphi_1+\ldots+\varphi_n,\hat{\varphi}_0+\varphi_1+\ldots+\varphi_n).$$

where for $n\in\mathbb{N}$ we used again the notations $\underline{\varphi}:=(\varphi_1,\ldots,\varphi_n)$ and (3.1.15).

PROOF. Using the same steps as in the proof of Proposition 3.13 we get for $U_1=\mathbb{1}_{\mathcal{B}}$ and $U_2=\mathbb{1}_{\mathcal{A}}$

$$\omega_n(A)=\int F(d\varphi_1)\int F(d\varphi_0)a(\varphi_1,\varphi_0)\rho_n(\varphi_0,\varphi_1)$$

with a kernel ρ_n given by

$$\rho_n(\varphi_0,\hat{\varphi}_0) = \int F^n(d\underline{\varphi})\sum_{k\in\mathbb{N}}\overline{\mathcal{V}_n\Psi_k(\varphi_1,\ldots\varphi_{n-1},\hat{\varphi}_0,\varphi_n)}\mathcal{V}_n\Psi_k(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n).$$

From Proposition 3.6 there follows

$$\overline{\mathcal{V}_n\Psi_k(\varphi_1,\ldots\varphi_{n-1},\hat{\varphi}_0,\varphi_n)} = \overline{g_n(\varphi_1,\ldots\varphi_{n-1},\hat{\varphi}_0,\varphi_n)}\cdot\overline{\Psi_k(\varphi_1+\ldots+\varphi_n+\hat{\varphi}_0)}$$

and

$$\mathcal{V}_n\Psi_k(\varphi_1,\ldots\varphi_{n-1},\varphi_0,\varphi_n)=g_n(\varphi_1,\ldots\varphi_{n-1},\varphi_0,\varphi_n)\cdot\Psi_k(\varphi_0+\ldots+\varphi_n).$$

With (3.2.17) we have

$$\begin{aligned}\rho_n(\varphi_0,\hat\varphi_0) &= \int F^n(d\underline{\varphi})\sum_{k\in\mathbb{N}}\overline{g_n(\varphi_1,\ldots,\varphi_{n-1},\hat\varphi_0,\varphi_n)}\ \overline{\Psi_k(\varphi_1+\ldots+\varphi_n+\hat\varphi_0)}\\ &\quad\cdot g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\Psi_k(\varphi_0+\ldots+\varphi_n)\\ &= \int F^n(d\underline{\varphi})\overline{g_n(\varphi_1,\ldots,\varphi_{n-1},\hat\varphi_0,\varphi_n)}\cdot g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi_0,\varphi_n)\\ &\quad\cdot\ \rho(\varphi_0+\varphi_1+\ldots+\varphi_n,\hat\varphi_0+\varphi_1+\ldots+\varphi_n).\end{aligned}$$

□

3.3 The Chain of Position Distributions in the General Case

In [16] there was developed a relation between locally normal states and point processes using the concept of position distribution.
In contrast to classical systems the position distribution alone is not sufficient to characterize a state of a quantum system but it still contains a lot of information about the state.
In this section we will give a description of the position distribution of the states $\omega_{n]}$ and ω_n for the generalized splitting procedure considered in the two preceeding sections.

Let $\underline{Y}\in\mathfrak{M}^n$. The operator of multiplication by the indicator function $\chi_{\underline{Y}}$

$$O_{\underline{Y}}\Psi(\underline{\varphi}) := \chi_{\underline{Y}}(\underline{\varphi})\cdot\Psi(\underline{\varphi}) \qquad (\Psi\in\mathcal{M}^n,\ \underline{\varphi}\in M^n)$$

belongs to $\mathfrak{L}(\mathcal{M}^n)$.

Remark 3.15. *In what follows we assume again that $\mathcal{A}=\mathcal{B}=\mathfrak{L}(\mathcal{M})$. So, $\mathcal{A}=\mathcal{B}$ contains all multiplication operators O_Y for $Y\in\mathfrak{M}$.*
$\mathcal{B}^n$ contains $\mathfrak{O}^n := \{O_{\underline{Y}}\ ,\ \underline{Y}\in\mathfrak{M}^n\}$.

Remark 3.16. *If τ is a normal state on $\mathcal{A}$ then for $Y\in\mathfrak{M}$*

$$Q_\tau(Y) := \tau(O_Y)$$

defines a point process on $[M,\mathfrak{M}]$ with $Q_\tau \ll F$ and a Radon-Nikodym derivative $\kappa=\frac{dQ_\tau}{dF}$ is given by

$$\kappa(\varphi)=\rho(\varphi,\varphi) \qquad (\varphi\in M), \tag{3.3.1}$$

where ρ is a kernel of the density matrix of τ of the form (3.2.13)*. For more details see [17], [22].*

Definition 3.17. *Let ω be the homogenous quantum Markov chain associated to the initial state τ and the transition expectation $\mathcal{E}_{U_1,U_2}$. For all $n \geq 1$*

(a) the probability measure on $[M^n, \mathfrak{M}^n]$ defined by

$$Q_{n]}^{U_1,U_2}(\underline{Y}) := \omega_{n]}^{U_1,U_2}(O_{\underline{Y}}) \qquad (\underline{Y} \in \mathfrak{M}^n) \tag{3.3.2}$$

is called the POSITION DISTRIBUTION of the state $\omega_{n]}^{U_1,U_2}$.
We set $Q_{n]} := Q_{n]}^{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}$.

(b) the probability measure on $[M, \mathfrak{M}]$ defined by

$$Q_{n}^{U_1,U_2}(Y) := \omega_{n}^{U_1,U_2}(O_{Y}) \qquad (Y \in \mathfrak{M}) \tag{3.3.3}$$

is called the POSITION DISTRIBUTION of the state $\omega_{n}^{U_1,U_2}$.
We set $Q_{n} := Q_{n}^{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}$.

Corollary 3.18. *Let $n \in \mathbb{N}$ and U_1, U_2 be isometric operators on $\mathcal{M}$.*
Then we have for $\underline{Y} \in \mathfrak{M}^n$ and $Y \in \mathfrak{M}$

$$\begin{aligned} Q_{n]}^{U_1,U_2}(\underline{Y}) \;=\; & \int F(d\varphi_n) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_n}{\varphi_{n-1}} \chi_{\underline{Y}}(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}) \cdot \\ & \cdot\, \rho_{n]}^{U_1,U_2}(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}, \varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}) \end{aligned} \tag{3.3.4}$$

and

$$Q_{n}^{U_1,U_2}(Y) = \int\limits_{Y} F(d\varphi) \rho_{n}^{U_1,U_2}(\varphi, \varphi) \tag{3.3.5}$$

with $\rho_{n]}^{U_1,U_2}$ defined in (3.2.9) and $\rho_{n}^{U_1,U_2}$ defined in (3.2.15).

PROOF. This follows from the Definition 3.17 and the Propositions 3.10 and 3.13.
□

Example 3.4. *Let $n \in \mathbb{N}$ and $U_1 := O_{h_1}$, $U_2 := O_{h_2}$ be operators of multiplication on $\mathcal{M}$ with functions h_1 and h_2, respectively, where $h_j : M \longrightarrow \mathbb{C}$, $|h_j(\varphi)| = 1$ for all $\varphi \in M$, $j = 1, 2$.*
Then we have for $\underline{Y} \in \mathfrak{M}^n$

$$\begin{aligned} Q_{n]}^{O_{h_1},O_{h_2}}(\underline{Y}) = & \int F(d\varphi_n) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_n}{\varphi_{n-1}} \chi_{\underline{Y}}(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}) \cdot \\ & \cdot\, \rho_{n]}(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}, \varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}) \\ = & \int F(d\varphi_{n+1}) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_{n+1}} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_{n+1}}{\varphi_{n}} \chi_{\underline{Y}}(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_n - \varphi_{n-1}) \cdot \\ & \cdot |g_n(\varphi_1, \varphi_2 - \varphi_1, \ldots, \varphi_{n+1} - \varphi_n)|^2 \cdot \kappa(\varphi_{n+1}) \end{aligned}$$

with
$$\rho_{n]}(\underline{\varphi},\underline{\varphi}) = \int F(d\varphi)|g_n(\underline{\varphi},\varphi)|^2 \cdot \kappa(\varphi+\varphi_1+\ldots+\varphi_n),$$
where we used the notations $\underline{\varphi} := (\varphi_1,\ldots,\varphi_n)$*,* $\varphi_i \in M$ *for all* $i \in n]$*,* (3.1.15) *and* (3.3.1).

PROOF. Using Corollary 3.18 and part (d) of Proposition 2.11 we get

$$\begin{aligned}
Q_{n]}^{O_{h_1},O_{h_2}}(\underline{Y}) &= \int F(d\varphi_n)\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n}\binom{\varphi_2}{\varphi_1}\cdots\binom{\varphi_n}{\varphi_{n-1}}\chi_{\underline{Y}}(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\cdot\\
&\quad\cdot\rho_{n]}^{O_{h_1},O_{h_2}}(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1},\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\\
&= \int F(d\varphi_n)\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n}\binom{\varphi_2}{\varphi_1}\cdots\binom{\varphi_n}{\varphi_{n-1}}\chi_{\underline{Y}}(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\int F(d\varphi)|h_1(\varphi_1)|^2\cdot\\
&\quad\cdot\prod_{i=1}^{n-1}|h_1(\varphi_{i+1}-\varphi_i)|^2|g_n(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1},\varphi)|^2\cdot\rho(\varphi_n+\varphi,\varphi_n+\varphi)\\
&= \int F(d\varphi_n)\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n}\binom{\varphi_2}{\varphi_1}\cdots\binom{\varphi_n}{\varphi_{n-1}}\chi_{\underline{Y}}(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\\
&\quad\int F(d\varphi)|g_n(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1},\varphi)|^2\cdot\rho(\varphi_n+\varphi,\varphi_n+\varphi)\\
&= \int F(d\varphi_{n+1})\sum_{\varphi_1\subseteq\ldots\subseteq\varphi_{n+1}}\binom{\varphi_2}{\varphi_1}\cdots\binom{\varphi_{n+1}}{\varphi_n}\chi_{\underline{Y}}(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_n-\varphi_{n-1})\cdot\\
&\quad|g_n(\varphi_1,\varphi_2-\varphi_1,\ldots,\varphi_{n+1}-\varphi_n)|^2\cdot\kappa(\varphi_{n+1}).
\end{aligned}\tag{3.3.6}$$

□

Remark 3.19. *For* $Q_{n]}^{U_1,U_2}$ *with* $U_1 = \mathbb{1}_{\mathcal{B}}$ *and* $U_2 = \mathbb{1}_{\mathcal{A}}$ *we get the same result as for* $U_1 = O_{h_1}$ *and* $U_2 = O_{h_2}$ *in Example 3.4.*

Example 3.5. *Let* $n \in \mathbb{N}$ *and* $U_1 := O_{h_1}$*,* $U_2 := O_{h_2}$ *be the operators of multiplication on* $\mathcal{M}$ *with a function* h_1 *and* h_2*, respectively,* $h_j : M \longrightarrow \mathbb{C}$ *,* $|h_j(\varphi)| = 1$ *for all* $\varphi \in M$*,* $j = 1,2$*. Furthermore, let* $Q_n^{U_1,U_2}$ *be given by Definition 3.17.*
Then we have for $Y \in \mathfrak{M}$

$$\begin{aligned}
Q_n^{O_{h_1},O_{h_2}}(Y) = Q_n(Y) &= \int_Y F(d\varphi)\rho_n(\varphi,\varphi)\\
&= \int_Y F(d\varphi)\int F^n(d\underline{\varphi})|g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi,\varphi_n)|^2\kappa(\sum_{i=1}^n\varphi_i+\varphi),
\end{aligned}\tag{3.3.7}$$

where we used the notations $Q_n := Q_n^{\mathbb{1}_{\mathcal{B}},\mathbb{1}_{\mathcal{A}}}$ *,* $\underline{\varphi} := (\varphi_1,\ldots,\varphi_n)$*,* $\varphi_i \in M$ *for all* $i \in n]$*,* (3.1.15) *and* (3.3.1).

PROOF. This follows directly from Corollary 3.18 and Example 3.3. □

We want to give further characterization of the position distribution at time n. First we need some preparations.

Lemma 3.20. *For $\varphi \in M$, $Y \in \mathfrak{M}$ let*

$$H_1(\varphi, Y) := \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \chi_Y(\widehat{\varphi}) |g(\widehat{\varphi}, \varphi - \widehat{\varphi})|^2, \tag{3.3.8}$$

$$H_n(\varphi, Y) := \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} |g(\widehat{\varphi}, \varphi - \widehat{\varphi})|^2 \cdot H_{n-1}(\varphi - \widehat{\varphi}, Y) \quad (n \geq 2). \tag{3.3.9}$$

Then for all $n \in \mathbb{N}$

$$H_n(\varphi, Y) = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \sum_{\varphi_2 \subseteq \varphi - \varphi_1} \binom{\varphi - \varphi_1}{\varphi_2} \cdots \sum_{\varphi_n \subseteq \varphi - \varphi_1 - \ldots - \varphi_{n-1}} \binom{\varphi - \varphi_1 - \ldots - \varphi_{n-1}}{\varphi_n} \cdot$$
$$\cdot \chi_Y(\varphi - \varphi_1 - \ldots - \varphi_n) |g_n(\varphi_1, \ldots, \varphi_{n-1}, \varphi - \varphi_1 - \ldots - \varphi_n, \varphi_n)|^2. \tag{3.3.10}$$

PROOF. For $n = 1$ we have because of part (a) of Lemma 2.8

$$H_1(\varphi, Y) = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_Y(\varphi_1) |g(\varphi_1, \varphi - \varphi_1)|^2 = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_Y(\varphi - \varphi_1) |g(\varphi - \varphi_1, \varphi_1)|^2.$$

Now assume that (3.3.10) holds for all $k < n$.
By definition of g_n in (3.1.15) there holds

$$g_n(\varphi_0, \varphi_1, \ldots, \varphi_n) = g(\varphi_0, \varphi_1 + \ldots + \varphi_n) \cdot g_{n-1}(\varphi_1, \varphi_2, \ldots, \varphi_n)$$

and thus because of $\varphi_2 + \ldots + \varphi_{n-1} + (\varphi - \varphi_1 - \ldots - \varphi_n) + \varphi_n = \varphi - \varphi_1$ also

$$g_n(\varphi_1, \ldots, \varphi_{n-1}, \varphi - \varphi_1 - \ldots - \varphi_n, \varphi_n)$$
$$= g(\varphi_1, \varphi - \varphi_1) \cdot g_{n-1}(\varphi_2, \ldots, \varphi_{n-1}, \varphi - \varphi_1 - \ldots - \varphi_n, \varphi_n).$$

Hence we get

$$H_n(\varphi, Y) = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} |g(\varphi_1, \varphi - \varphi_1)|^2 \cdot H_{n-1}(\varphi - \varphi_1, Y)$$
$$= \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} |g(\varphi_1, \varphi - \varphi_1)|^2 \sum_{\varphi_2 \subseteq \varphi - \varphi_1} \binom{\varphi - \varphi_1}{\varphi_2} \cdots \sum_{\varphi_n \subseteq \varphi - \varphi_1 - \ldots - \varphi_{n-1}} \binom{\varphi - \varphi_1 - \ldots - \varphi_{n-1}}{\varphi_n}$$
$$\chi_Y(\varphi - \varphi_1 - \ldots - \varphi_n) |g_{n-1}(\varphi_2, \ldots, \varphi_{n-1}, \varphi - \varphi_1 - \ldots - \varphi_n, \varphi_n)|^2$$
$$= \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \sum_{\varphi_2 \subseteq \varphi - \varphi_1} \binom{\varphi - \varphi_1}{\varphi_2} \cdots \sum_{\varphi_n \subseteq \varphi - \varphi_1 - \ldots - \varphi_{n-1}} \binom{\varphi - \varphi_1 - \ldots - \varphi_{n-1}}{\varphi_n}$$
$$\chi_Y(\varphi - \varphi_1 - \ldots - \varphi_n) |g_n(\varphi_1, \ldots, \varphi_{n-1}, \varphi - \varphi_1 - \ldots - \varphi_n, \varphi_n)|^2.$$

So, (3.3.10) holds for all $n \in \mathbb{N}$. □

All H_n are stochastic kernels, because they are measurable with respect to the first component and for fixed $\varphi \in M$, $H_n(\varphi, \cdot)$ is a probability measure on $[M, \mathfrak{M}]$, i.e. a point process.

Proposition 3.21. *For all $n \geq 1$ and $Y \in \mathfrak{M}$ there holds*

$$Q_n(Y) = \int Q_\tau(d\varphi) H_n(\varphi, Y) \tag{3.3.11}$$

with Q_τ being the position distribution of τ and H_n the stochastic kernel given by (3.3.8) *and* (3.3.9)*.*

PROOF. Let $n = 1$. From (3.3.7) and (3.3.10) we obtain using part (b) of Proposition 2.11

$$\begin{aligned}
Q_1(Y) &= \int_Y F(d\varphi)\rho_1(\varphi,\varphi) = \int F(d\varphi)\chi_Y(\varphi) \int F(d\varphi_1) |g(\varphi,\varphi_1)|^2 \kappa(\varphi + \varphi_1) \\
&= \int F(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_Y(\varphi - \varphi_1) |g(\varphi - \varphi_1, \varphi_1)|^2 \kappa(\varphi) \\
&= \int Q_\tau(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_Y(\varphi_1) |g(\varphi_1, \varphi - \varphi_1)|^2 = \int Q_\tau(d\varphi) H_1(\varphi, Y).
\end{aligned}$$

Now let $n \geq 2$. From (3.3.7) and (3.3.10) we obtain using part (b) of Proposition 2.11 n times

$$\begin{aligned}
Q_n(Y) &= \int_Y F(d\varphi)\rho_n(\varphi,\varphi) \\
&= \int F(d\varphi)\chi_Y(\varphi) \int F(d\varphi_1) \dots \int F(d\varphi_n) \\
&\qquad\qquad |g_n(\varphi_1, \dots, \varphi_{n-1}, \varphi, \varphi_n)|^2 \kappa(\varphi + \sum_{i=1}^{n} \varphi_i) \\
&= \int F(d\varphi) \sum_{\varphi_n \subseteq \varphi} \binom{\varphi}{\varphi_n} \chi_Y(\varphi - \varphi_n) \int F(d\varphi_1) \dots \int F(d\varphi_{n-1}) \\
&\qquad\qquad |g_n(\varphi_1, \dots, \varphi_{n-1}, \varphi - \varphi_n, \varphi_n)|^2 \kappa(\varphi + \sum_{i=1}^{n-1} \varphi_i) \\
&= \int F(d\varphi) \sum_{\varphi_{n-1} \subseteq \varphi} \binom{\varphi}{\varphi_{n-1}} \sum_{\varphi_n \subseteq \varphi} \binom{\varphi}{\varphi_n} \chi_Y(\varphi - \varphi_n - \varphi_{n-1}) \int F(d\varphi_1) \dots \\
&\qquad \dots \int F(d\varphi_{n-2}) |g_n(\varphi_1, \dots, \varphi_{n-1}, \varphi - \varphi_n - \varphi_{n-1}, \varphi_n)|^2 \kappa(\varphi + \sum_{i=1}^{n-2} \varphi_i) \\
&= \int F(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \sum_{\varphi_2 \subseteq \varphi - \varphi_1} \binom{\varphi - \varphi_1}{\varphi_2} \dots \sum_{\varphi_n \subseteq \varphi - \varphi_1 - \dots - \varphi_{n-1}} \binom{\varphi - \varphi_1 - \dots - \varphi_{n-1}}{\varphi_n}
\end{aligned}$$

$$\cdot\chi_Y(\varphi-\varphi_1-\ldots-\varphi_n)|g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi-\varphi_1-\ldots-\varphi_n,\varphi_n)|^2\kappa(\varphi)$$
$$= \int Q_\tau(d\varphi)\sum_{\varphi_1\subseteq\varphi}\binom{\varphi}{\varphi_1}\sum_{\varphi_2\subseteq\varphi-\varphi_1}\binom{\varphi-\varphi_1}{\varphi_2}\ldots\sum_{\varphi_n\subseteq\varphi-\varphi_1-\ldots-\varphi_{n-1}}\binom{\varphi-\varphi_1-\ldots-\varphi_{n-1}}{\varphi_n}$$
$$\cdot\chi_Y(\varphi-\varphi_1-\ldots-\varphi_n)|g_n(\varphi_1,\ldots,\varphi_{n-1},\varphi-\varphi_1-\ldots-\varphi_n,\varphi_n)|^2$$
$$= \int Q_\tau(d\varphi)H_n(\varphi,Y).$$

□

3.4 Geometric Splitting

Now we will give an example of a splitting function g for a non-independent splitting. It was introduced in [27] for $\mathcal{V}$ defined on the ℓ^2 space over the natural numbers which is canonically isomorphic to the symmetric Fock space over $\mathbb{C}$. In section 7.3 of [32] there were discussed independence questions concerning this splitting on the symmetric Fock space over general G.
We will also consider the geometric splitting on the symmetric Fock space over G and describe the corresponding position distribution at time n. Especially for $n=1$ the position distribution has an interesting form.

Proposition 3.22. *Let q be a constant with $q\in(0,1)$ and $p:=1-q$. Then the function $g: M\times M\longrightarrow\mathbb{C}$ defined by*

$$g(\varphi_1,\varphi_2) := \sqrt{\frac{1}{\binom{|\varphi_1+\varphi_2|}{|\varphi_1|}}\cdot p^{\Delta(|\varphi_2|)}\cdot q^{|\varphi_1|}} \tag{3.4.1}$$

with $\Delta:\mathbb{N}_0\longrightarrow\{0,1\}$,

$$\Delta(n) := \begin{cases} 1 & \text{for } n>0,\\ 0 & \text{for } n=0\end{cases} \tag{3.4.2}$$

satisfies the isometry condition (3.1.1) *for all $\varphi\in M$.*

PROOF. First let $\varphi=\mathfrak{o}$. Then we have

$$\sum_{\widehat{\varphi}\subseteq\varphi}\binom{\varphi}{\widehat{\varphi}}|g(\widehat{\varphi},\varphi-\widehat{\varphi})|^2 = (\sqrt{q^0})^2 = 1. \tag{3.4.3}$$

Now let $\varphi\in M$, $|\varphi|=n>0$. Then we get using Corollary 2.10

$$\sum_{\widehat{\varphi}\subseteq\varphi}\binom{\varphi}{\widehat{\varphi}}|g(\widehat{\varphi},\varphi-\widehat{\varphi})|^2 = \sum_{\widehat{\varphi}\subset\varphi}\binom{\varphi}{\widehat{\varphi}}\frac{1}{\binom{n}{|\widehat{\varphi}|}}\cdot p\cdot q^{|\widehat{\varphi}|}+q^n = \sum_{k=0}^{n-1}\binom{n}{k}\cdot\frac{1}{\binom{n}{k}}\cdot p\cdot q^k+q^n$$
$$= p\cdot\sum_{k=0}^{n-1}q^k+q^n = p\cdot\frac{1-q^n}{1-q}+q^n = 1.$$

□

Remark 3.23. *It is obvious that the definition of the function g in* (3.4.1) *is equivalent to*

$$g(\varphi_1, \varphi_2) = \begin{cases} \sqrt{q^{|\varphi_1|}} & \text{for } \varphi_2 = \mathfrak{o}, \\ \sqrt{\frac{1}{\binom{|\varphi_1+\varphi_2|}{|\varphi_1|}} \cdot p \cdot q^{|\varphi_1|}} & \text{for } \varphi_2 \neq \mathfrak{o}. \end{cases} \tag{3.4.4}$$

Because of Proposition 3.22 and Lemma 3.1 $\mathcal{V} : \mathcal{M} \longrightarrow \mathcal{M}^2$ defined on $\mathcal{M}$ by

$$\mathcal{V}\ \Psi(\varphi_1, \varphi_2) := g(\varphi_1, \varphi_2) \cdot \Psi(\varphi_1 + \varphi_2) \qquad (\Psi \in \mathcal{M}\ ,\ \varphi_1, \varphi_2 \in M)$$

is an isometry.
Now we consider a sequence of n generalized splitting procedures described by g.

Proposition 3.24. *Let for $n \in \mathbb{N}$ the function $g_n : M^{n+1} \longrightarrow \mathbb{C}$ be defined by* (3.1.15) *with g given by* (3.4.1), *$g_0 := 1$. Then it holds*

$$g_n(\varphi_0, \ldots, \varphi_n) = \begin{cases} g_{n-1}(\varphi_0, \ldots, \varphi_{n-1}) \cdot \sqrt{q^{|\varphi_{n-1}|}} & \text{for } \varphi_n = \mathfrak{o}, \\ \sqrt{\frac{|\varphi_0|! \cdot \ldots \cdot |\varphi_n|!}{|\varphi_0 + \ldots + \varphi_n|!} \cdot p^n \cdot q^{|\varphi_0 + \ldots + \varphi_{n-1}|}} & \text{for } \varphi_n \neq \mathfrak{o} \end{cases} \tag{3.4.5}$$

$$= \sqrt{\frac{|\varphi_0|! \cdot \ldots \cdot |\varphi_n|!}{|\varphi_0 + \ldots + \varphi_n|!} \cdot p^{\Delta(|\varphi_1 + \ldots + \varphi_n|) + \ldots + \Delta(|\varphi_{n-1} + \varphi_n|) + \Delta(|\varphi_n|)} \cdot q^{|\varphi_0 + \ldots + \varphi_{n-1}|}}, \tag{3.4.6}$$

where again Δ is defined by (3.4.2).

PROOF. First let $\varphi_n = \mathfrak{o}$. Then we have

$$g_n(\varphi_0, \ldots, \varphi_n) = \prod_{k=0}^{n-1} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n)$$
$$= \prod_{k=0}^{n-2} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_{n-1}) \cdot \sqrt{q^{|\varphi_{n-1}|}} = g_{n-1}(\varphi_0, \ldots, \varphi_{n-1}) \cdot \sqrt{q^{|\varphi_{n-1}|}}.$$

Now let $\varphi_n \neq \mathfrak{o}$. This implies that in the product $\prod_{k=0}^{n-1} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n)$ all second arguments of g are different from $\mathfrak{o}$. Hence we have

$$g_n(\varphi_0, \ldots, \varphi_n) = \prod_{k=0}^{n-1} g(\varphi_k, \varphi_{k+1} + \ldots + \varphi_n)$$
$$= \sqrt{\frac{1}{\binom{|\varphi_0 + \ldots + \varphi_n|}{|\varphi_0|}} \cdot pq^{|\varphi_0|} \cdot \frac{1}{\binom{|\varphi_1 + \ldots + \varphi_n|}{|\varphi_1|}} \cdot pq^{|\varphi_1|} \cdot \ldots \cdot \frac{1}{\binom{|\varphi_{n-1} + \varphi_n|}{|\varphi_{n-1}|}} \cdot pq^{|\varphi_{n-1}|}}$$
$$= \sqrt{\frac{|\varphi_0|! \cdot \ldots \cdot |\varphi_n|!}{|\varphi_0 + \ldots + \varphi_n|!} \cdot p^n \cdot q^{|\varphi_0 + \ldots + \varphi_{n-1}|}}.$$

Applying the same procedure to all $k \in n]$ we get (3.4.6). $\square$

Now we give a formula for the position distribution of the geometric splitting.

Corollary 3.25. *Let g be defined by* (3.4.1) *and let for $n \in \mathbb{N}$ the position distribution Q_n at time n be given by* (3.3.7)*. Then, according to Proposition 3.21, there holds*

$$Q_n(Y) = \int Q_\tau(d\varphi) H_n(\varphi, Y)$$

with

$$H_1(\varphi, Y) = \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \chi_Y(\widehat{\varphi}) \frac{1}{\binom{|\varphi|}{|\widehat{\varphi}|}} \cdot p^{\Delta(|\varphi - \widehat{\varphi}|)} \cdot q^{|\widehat{\varphi}|}, \tag{3.4.7}$$

$$H_n(\varphi, Y) = \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \frac{1}{\binom{|\varphi|}{|\widehat{\varphi}|}} p^{\Delta(|\varphi - \widehat{\varphi}|)} \cdot q^{|\widehat{\varphi}|} \cdot H_{n-1}(\varphi - \widehat{\varphi}, Y). \tag{3.4.8}$$

PROOF. This follows directly from (3.4.1), (3.3.9) and (3.3.8).

□

For $\varphi \in M$ and $B \in \mathfrak{G}$ we denote by $\varphi_{|_B}$ the restriction of φ to $\mathfrak{G} \cap B$, i.e. $\varphi_{|_B}(.) = \varphi(. \cap B)$. From (3.4.8) one gets for all $B \in \mathfrak{B}$, $k \in \mathbb{N}$ and $\varphi \in M^f$ the following representation of H_1 as a composition of geometric and hypergeometric distribution.

Corollary 3.26. *Let $\varphi \in M$ with $|\varphi| =: l \in \mathbb{N}$, $B \in \mathfrak{B}$ and $|\varphi_{|_B}| =: m$. Then for $k \in \mathbb{N}$ there holds*

$$H_1(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) = \begin{cases} \sum_{i=0}^{l-m} \mathbb{H}_{l,m,i+k}(k) \cdot \mathbb{G}_p(i+k) & \text{for } 0 \le k \le m-1 \\ q^l + \sum_{i=0}^{l-m-1} \mathbb{H}_{l,m,i+m}(m) \cdot \mathbb{G}_p(i+m) & \text{for } k = m \\ 0 & \text{else,} \end{cases} \tag{3.4.9}$$

where $\mathbb{G}_p(k) := p \cdot q^k$ defined for $k \in \mathbb{N}_0$, $p \in (0,1)$, $q := 1-p$ is the geometric distribution and $\mathbb{H}_{N,m,n}(k) := \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$ defined for $N, m, n \in \mathbb{N}$, $m \le N$, $n \le N$, $k \in \{0, \dots, max(m,n)\}$ is the hypergeometric distribution.

PROOF. According to the definition of H_1 in (3.3.8) we get using Lemma 2.8

$$\begin{aligned} H_1(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) &= \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_{\{\widetilde{\varphi}:\widetilde{\varphi}(B)=k\}}(\varphi_1) \cdot |g(\varphi_1, \varphi - \varphi_1)|^2 \\ &= \sum_{\substack{\varphi_1 \subseteq \varphi_{|_B} \\ |\varphi_1| = k}} \sum_{\varphi_2 \subseteq \varphi_{|_{B^C}}} \binom{\varphi_{|_B} + \varphi_{|_{B^C}}}{\varphi_1 + \varphi_2} |g(\varphi_1 + \varphi_2, \varphi - \varphi_1 - \varphi_2)|^2 \\ &= \sum_{\substack{\varphi_1 \subseteq \varphi_{|_B} \\ |\varphi_1| = k}} \binom{\varphi_{|_B}}{\varphi_1} \sum_{\varphi_2 \subseteq \varphi_{|_{B^C}}} \binom{\varphi_{|_{B^C}}}{\varphi_2} |g(\varphi_1 + \varphi_2, (\varphi_{|_B} - \varphi_1) + (\varphi_{|_{B^C}} - \varphi_2))|^2 \end{aligned}$$

$$= \binom{m}{k} \cdot \sum_{i=0}^{l-m} \binom{l-m}{i} \frac{1}{\binom{l}{k+i}} \cdot p^{\Delta(l-k-i)} \cdot q^{k+i}$$

$$= \sum_{i=0}^{l-m} \frac{\binom{m}{k}\binom{l-m}{i}}{\binom{l}{k+i}} \cdot p^{\Delta(l-k-i)} \cdot q^{k+i}. \tag{3.4.10}$$

Because of (3.4.2) $p^{\Delta(l-k-i)}$ in (3.4.10) is different from zero only for $0 \leq k \leq m$. Now we have to consider two cases. For $0 \leq k \leq m-1$ there holds $l > k+i$ for all summands in (3.4.10). Hence in this case there holds

$$H_1(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) = \sum_{i=0}^{l-m} \frac{\binom{m}{k}\binom{l-m}{i}}{\binom{l}{i+k}} \cdot p \cdot q^{i+k} = \sum_{i=0}^{l-m} \mathbb{H}_{l,m,i+k}(k) \cdot \mathbb{G}_p(i+k). \tag{3.4.11}$$

In the second case for $k = m$ in (3.4.10) $\Delta(l-k-i)$ becomes zero only for $i = l-m$. Hence

$$H_1(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) = \sum_{i=0}^{l-m-1} \frac{\binom{l-m}{i}}{\binom{l}{i+m}} \cdot p \cdot q^{i+m} + q^{(l-m)+m}$$

$$= q^l + \sum_{i=0}^{l-m-1} \frac{\binom{l-m}{i}}{\binom{l}{i+m}} \cdot p \cdot q^{i+m} = q^l + \sum_{i=0}^{l-m-1} \mathbb{H}_{l,m,i+m}(m) \cdot \mathbb{G}_p(i+m).$$

This completes the proof. $\square$

Chapter 4

Independent Beam Splitting

4.1 Definition and Basic Properties

We now consider so-called independent beam splittings. This splitting procedure turns one input beam into two output beams, one reflected (or absorbed or destroyed) and one transmitted.

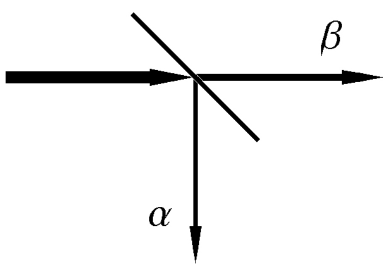

Figure 1 : Independent beam splitting

In our model this means for given reflection rate $|\alpha|^2$ and transmission rate $|\beta|^2$ each point of the original configuration chooses independently from all other points with probability $|\alpha|^2$ one subconfiguration and with probability $|\beta|^2$ the other one.
Independent beam splitting models with constant splitting rates for quantum systems were for instance considered in [2] and [38]. In [24] these models were generalized to locally normal states on a quasilocal algebra over a symmetric Fock space, in [19], [26], [33], [32] there were used complex-valued functions α and β to define the splitting rates.
In the sequel we will condense the results from these works and give proofs omitted there. Others are presented more detailed or in a more general form.
In some points we will expand the results by adding independent evolutions of the quantum system and the measurement apparatus as it was prepared in chapter 3.

Let again $\mathcal{A} = \mathcal{B} = \mathfrak{L}(\mathcal{M})$.

Proposition 4.1. *Let* $g(\varphi_1, \varphi_2) := \mathbb{e}_\alpha\ (\varphi_1) \cdot \mathbb{e}_\beta\ (\varphi_2)$ *with measurable functions* $\alpha, \beta : G \longrightarrow \mathbb{C}$. *The isometry condition* (3.1.1) *is then equivalent to*

$$|\alpha(x)|^2 + |\beta(x)|^2 = 1 \qquad \forall\ x \in G. \tag{4.1.1}$$

The isometries $\mathcal{V}$ bzw. $\mathcal{V}^*$ from section 3.1 have the form

$$\mathcal{V}_{\alpha,\beta}\Psi(\varphi_1,\varphi_2) = \mathbb{e}_{\alpha}\ (\varphi_1)\mathbb{e}_{\beta}\ (\varphi_2)\Psi(\varphi_1+\varphi_2) \tag{4.1.2}$$

for all $\Psi \in \mathcal{M}, \varphi_1, \varphi_2 \in M$, and

$$\mathcal{V}^*_{\alpha,\beta}\Phi(\varphi) = \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} \mathbb{e}_{\overline{\alpha}}\ (\hat{\varphi})\mathbb{e}_{\overline{\beta}}\ (\varphi-\hat{\varphi})\Phi(\hat{\varphi},\varphi-\hat{\varphi}) \tag{4.1.3}$$

for $\Phi \in \mathcal{M}\otimes\mathcal{M}, \varphi \in M$.
The corresponding transition expectation is

$$\mathcal{E}_{\alpha,\beta} := \mathcal{V}^*_{\alpha,\beta}(.)\mathcal{V}_{\alpha,\beta}. \tag{4.1.4}$$

PROOF. Using Definition 2.12 and (2.3.3) the isometry condition (3.1.1) for $g(\widehat{\varphi},\varphi-\widehat{\varphi}) = \mathbb{e}_{\alpha}\ (\widehat{\varphi})\cdot\mathbb{e}_{\beta}\ (\varphi-\widehat{\varphi})$ transforms to

$$\begin{aligned}\sum_{\widehat{\varphi}\subseteq\varphi}\binom{\varphi}{\widehat{\varphi}}|\mathbb{e}_{\alpha}(\widehat{\varphi})\cdot\mathbb{e}_{\beta}(\varphi-\widehat{\varphi})|^2 &= \sum_{\widehat{\varphi}\subseteq\varphi}\binom{\varphi}{\widehat{\varphi}}\mathbb{e}_{|\alpha|^2}(\widehat{\varphi})\cdot\mathbb{e}_{|\beta|^2}(\varphi-\widehat{\varphi}) \\ &= \mathbb{e}_{|\alpha|^2+|\beta|^2}(\varphi). \end{aligned}\tag{4.1.5}$$

$\mathbb{e}_{|\alpha|^2+|\beta|^2}(\varphi) = 1$ holds for F-almost all $\varphi \in M$ if and only if $|\alpha(x)|^2+|\beta(x)|^2 = 1$ for all $x \in G$.
The other formulae follow directly from (3.1.2) and Lemma 3.1.

□

Remark 4.2. *For an exponential vector $\mathbb{e}_h \in \mathcal{M}$, $\mathcal{V}_{\alpha,\beta}$ has the form*

$$\mathcal{V}_{\alpha,\beta}\mathbb{e}_h = \mathbb{e}_{\alpha h}\otimes\mathbb{e}_{\beta h}.$$

If we interpret $\mathbb{e}_h$ as a coherent beam, the tensor product structure of $\mathcal{V}$ indicates that this beam is split into two <u>*independent*</u>*, also coherent beams $\mathbb{e}_{\alpha h}$ and $\mathbb{e}_{\beta h}$ of lower intensity. $\|h\|^2$ can be interpreted as intensity of the beam. The condition $|\alpha|^2+|\beta|^2 = 1$ means that no loss is caused by the splitting.*

With additional independent evolutions of the measurement apparatus and the quantum system, described by isometric operators $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{A}$ we have for $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\mathcal{E}_{\alpha,\beta,U_1,U_2}(B\otimes A) := \mathcal{V}^*_{\alpha,\beta}(U_1^*BU_1\otimes U_2^*AU_2)\mathcal{V}_{\alpha,\beta}. \tag{4.1.6}$$

Remark 4.3. *The above defined mapping $\mathcal{E}_{\alpha,\beta,U_1,U_2}$ is of the form*

$$\mathcal{E}_{\alpha,\beta,U_1,U_2} = \mathcal{V}^*_{\alpha,\beta,U_1,U_2}\ (\cdot)\ \mathcal{V}_{\alpha,\beta,U_1,U_2} \tag{4.1.7}$$

with

$$\mathcal{V}_{\alpha,\beta,U_1,U_2} := (U_1\otimes U_2)\mathcal{V}_{\alpha,\beta} \quad \text{and} \quad \mathcal{V}^*_{\alpha,\beta,U_1,U_2} := \mathcal{V}^*_{\alpha,\beta}(U_1^*\otimes U_2^*).$$

$\mathcal{E}_{\alpha,\beta,\ U_1,U_2}$ fulfills the property (CP1) according to Definition 1.2, and is therefore a transition expectation.

PROOF. This follows immediately from Proposition 3.2. $\square$

For operators U_1, U_2 being second quantizations (see Definition 2.14) of isometries $\mathbf{v}_1$ and $\mathbf{v}_2$ $\mathcal{V}_{\alpha,\beta,U_1,U_2}$ and $\mathcal{V}^*_{\alpha,\beta,U_1,U_2}$ look as follows.

Proposition 4.4. *Let $\mathcal{V}_{\alpha,\beta,\ U_1,U_2}$ be defined as in Remark 4.3 with $U_1 = \Gamma(\mathbf{v}_1)$ and $U_2 = \Gamma(\mathbf{v}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are isometries from $\mathfrak{L}(\mathcal{L}^2(G,\nu))$. Then we have on exponential vectors $\mathbb{e}_f$, $\mathbb{e}_h \in \mathcal{M}$*

$$\begin{aligned} \mathcal{V}_{\alpha,\beta,\Gamma(\mathbf{v}_1),\Gamma(\mathbf{v}_2)}\mathbb{e}_h &= \mathbb{e}_{\mathbf{v}_1(\alpha h)} \otimes \mathbb{e}_{\mathbf{v}_2(\beta h)} \quad \textit{and} &(4.1.8)\\ \mathcal{V}^*_{\alpha,\beta,\Gamma(\mathbf{v}_1),\Gamma(\mathbf{v}_2)}\mathbb{e}_f \otimes \mathbb{e}_h &= \mathbb{e}_{\overline{\alpha}\mathbf{v}_1^* f + \overline{\beta}\mathbf{v}_2^* h}. &(4.1.9)\end{aligned}$$

For all $\Psi \in \mathcal{M} \otimes \mathcal{M}$, φ_1, $\varphi_2 \in M$

$$\mathcal{V}_{\alpha,\beta,\Gamma(\mathbf{v}_1),\Gamma(\mathbf{v}_2)}\Psi(\varphi_1,\varphi_2) = \mathbb{e}_{\mathbf{v}_1\alpha}\ (\varphi_1)\mathbb{e}_{\mathbf{v}_2\beta}\ (\varphi_2)\Psi(\varphi_1+\varphi_2), \tag{4.1.10}$$

and for all $\Phi \in \mathcal{M}$ and $\varphi \in M$

$$\mathcal{V}^*_{\alpha,\beta,\Gamma(\mathbf{v}_1),\Gamma(\mathbf{v}_2)}\Phi(\varphi) = \sum_{\hat{\varphi}\subseteq\varphi} \binom{\varphi}{\hat{\varphi}} \mathbb{e}_{\overline{\alpha}\mathbf{v}_1^*}\ (\hat{\varphi})\mathbb{e}_{\overline{\beta}\mathbf{v}_2^*}\ (\varphi-\hat{\varphi})\Phi(\hat{\varphi},\varphi-\hat{\varphi}). \tag{4.1.11}$$

Because $\mathbf{v}_1$, $\mathbf{v}_2$ are isometries, the condition (3.1.1) *corresponds in this case again to*

$$|\alpha(x)|^2 + |\beta(x)|^2 = 1 \qquad \forall\ x \in G. \tag{4.1.12}$$

PROOF. These statements follow from Remark 4.3, (2.3.5) and the fact that $\mathbf{v}_1$, $\mathbf{v}_2$ are isometries. $\square$

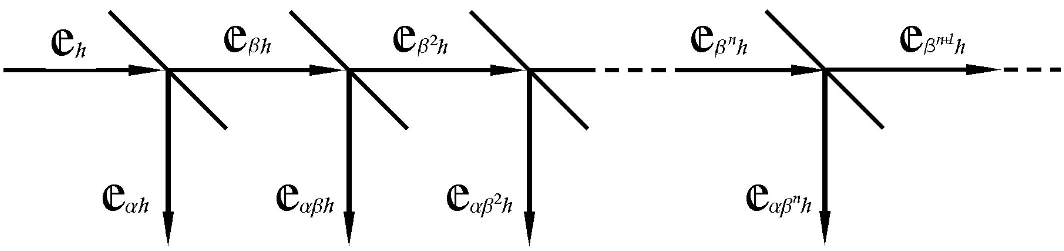

Figure 2 : Sequence of independent beam splittings

For a coherent inital state τ on $\mathcal{A}$ the states $\omega_n := \omega_n^{\mathbb{1}_\mathcal{B},\mathbb{1}_\mathcal{A}}$ on $\mathcal{B}$ and $\omega_{n]} := \omega_{n]}^{\mathbb{1}_\mathcal{B},\mathbb{1}_\mathcal{A}}$ on $\mathcal{B}^n$ defined in (3.2.1) and (3.2.2) look in the case of an independent beam splitting as follows.

Proposition 4.5. *Let τ be a coherent state on $\mathcal{A}$ corresponding to the function $h \in \mathcal{L}^2(G,\nu)$, i.e.*

$$\tau = e^{-\|h\|^2} \cdot \langle \mathbb{e}_h, \cdot\ \mathbb{e}_h \rangle_{\mathcal{M}}. \tag{4.1.13}$$

Then for all $n \in \mathbb{N}$ the state ω_n on $\mathcal{B}$ corresponding to τ and $\mathcal{E}_{\alpha,\beta}$ at time n is a coherent state given by

$$\omega_n = e^{-|\alpha|^2|\beta|^{2(n-1)}\|h\|^2} \cdot \langle \mathbb{e}_{\alpha\beta^{n-1}h}, .\mathbb{e}_{\alpha\beta^{n-1}h} \rangle_{\mathcal{M}}. \tag{4.1.14}$$

The state $\omega_{n]}$ up to time n on $\mathcal{B}^n$ corresponding to τ and $\mathcal{E}_{\alpha,\beta}$ is given by

$$\omega_{n]} = e^{-(1-|\beta|^{2n})\|h\|^2} \cdot \langle \mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n}, \cdot\ \mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n} \rangle_{\mathcal{M}^n} \tag{4.1.15}$$

with $h_i = \alpha\beta^{i-1}h$ for all $i \in n]$.

PROOF. From (4.1.13) and (3.2.3) it follows that

$$\rho(\varphi_0,\varphi_1) = e^{-\|h\|^2} \cdot \mathbb{e}_h(\varphi_0)\mathbb{e}_{\overline{h}}(\varphi_1) \tag{4.1.16}$$

is a kernel of the density matrix of τ. By putting the special choice of g and (4.1.16) into the result of Proposition 3.14 after applying part (d) of Proposition 2.11 we get

$$\begin{aligned}
\rho_n(\varphi_0,\widehat{\varphi}_0) = &\int F(d\varphi_n) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{\overline{\alpha}}(\widehat{\varphi}_0)\mathbb{e}_{\overline{\beta}}(\varphi_n - \varphi_{n-1})\mathbb{e}_\alpha(\varphi_0) \\
&\cdot \mathbb{e}_\beta(\varphi_n - \varphi_{n-1})\mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\beta(\varphi_0 + \varphi_n - \varphi_1)\mathbb{e}_{\overline{\alpha}}(\varphi_1)\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0 + \varphi_n - \varphi_1)\mathbb{e}_\alpha(\varphi_2 - \varphi_1) \\
&\cdot \mathbb{e}_\beta(\varphi_0 + \varphi_n - \varphi_2)\mathbb{e}_{\overline{\alpha}}(\varphi_2 - \varphi_1)\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0 + \varphi_n - \varphi_2)\ldots\mathbb{e}_\alpha(\varphi_{n-1} - \varphi_{n-2}) \\
&\cdot \mathbb{e}_\beta(\varphi_0 + \varphi_n - \varphi_{n-1})\mathbb{e}_{\overline{\alpha}}(\varphi_{n-1} - \varphi_{n-2})\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0 + \varphi_n - \varphi_{n-1}) \\
&\cdot e^{-\|h\|^2} \cdot \mathbb{e}_h(\varphi_0 + \varphi_n)\mathbb{e}_{\overline{h}}(\widehat{\varphi}_0 + \varphi_n).
\end{aligned} \tag{4.1.17}$$

For $\varphi_1 \subseteq \varphi_2 \subseteq \ldots \subseteq \varphi_{n-1}$ there holds

$$\mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\alpha(\varphi_2 - \varphi_1)\mathbb{e}_\alpha(\varphi_3 - \varphi_2)\ldots\mathbb{e}_\alpha(\varphi_{n-1} - \varphi_{n-2}) = \mathbb{e}_\alpha(\varphi_{n-1}). \tag{4.1.18}$$

Using this, we get from (4.1.17)

$$\begin{aligned}
\rho_n(\varphi_0,\widehat{\varphi}_0) = &\ \mathbb{e}_{\overline{\alpha}}(\widehat{\varphi}_0)\mathbb{e}_\alpha(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_\beta(\varphi_0)\Big)^{n-1}\int F(d\varphi_n) \sum_{\varphi_1\subseteq\ldots\subseteq\varphi_n} \binom{\varphi_2}{\varphi_1} \cdots \binom{\varphi_n}{\varphi_{n-1}} \\
&\cdot \mathbb{e}_{|\alpha|^2}(\varphi_{n-1}) \cdot \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_{n-1})\mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_1)\mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_2)\ldots\mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_{n-1}) \\
&\cdot e^{-\|h\|^2} \cdot \mathbb{e}_h(\varphi_0 + \varphi_n)\mathbb{e}_{\overline{h}}(\widehat{\varphi}_0 + \varphi_n) \\
= &\ \mathbb{e}_{\overline{\alpha}\overline{h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha h}(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_\beta(\varphi_0)\Big)^{n-1} \cdot e^{-\|h\|^2} \cdot \int F(d\varphi_n)\mathbb{e}_{|h|^2}(\varphi_n) \\
&\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2}(\varphi_{n-1})\mathbb{e}_{|\beta|^4}(\varphi_n - \varphi_{n-1}) \sum_{\varphi_{n-2}\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_{n-2})
\end{aligned}$$

$$\cdot \sum_{\varphi_{n-3}\subseteq\varphi_{n-2}} \dots \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_2) \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_1). \tag{4.1.19}$$

From

$$\mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_1) = \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_2)\cdot \mathbb{e}_{|\beta|^2}(\varphi_2-\varphi_1)$$

there follows

$$\begin{aligned} \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_1) &= \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_2)\mathbb{e}_{|\beta|^2}(\varphi_2-\varphi_1) \\ = \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_2) \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_2-\varphi_1) &= \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_2)\mathbb{e}_{1+|\beta|^2}(\varphi_2). \end{aligned} \tag{4.1.20}$$

If we insert this into (4.1.19), we get

$$\begin{aligned} \rho_n(\varphi_0,\widehat{\varphi}_0) &= \mathbb{e}_{\overline{\alpha}\overline{h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha h}(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_{\beta}(\varphi_0)\Big)^{n-1} \cdot e^{-\|h\|^2} \cdot \int F(d\varphi_n)\mathbb{e}_{|h|^2}(\varphi_n) \\ &\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2}(\varphi_{n-1})\mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_{n-1}) \sum_{\varphi_{n-2}\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_{n-2}) \\ &\cdot \sum_{\varphi_{n-3}\subseteq\varphi_{n-2}} \dots \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_2)\mathbb{e}_{1+|\beta|^2}(\varphi_2). \end{aligned} \tag{4.1.21}$$

From

$$\mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_2) = \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_3)\cdot \mathbb{e}_{|\beta|^4}(\varphi_3-\varphi_2)$$

there follows

$$\begin{aligned} &\sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_2)\mathbb{e}_{1+|\beta|^2}(\varphi_2) \\ &= \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_3)\mathbb{e}_{|\beta|^4}(\varphi_3-\varphi_2)\mathbb{e}_{1+|\beta|^2}(\varphi_2) \\ &= \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_3) \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \mathbb{e}_{|\beta|^4}(\varphi_3-\varphi_2)\mathbb{e}_{1+|\beta|^2}(\varphi_2) = \mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_3)\mathbb{e}_{1+|\beta|^2+|\beta|^4}(\varphi_3). \end{aligned}$$

If we insert this into (4.1.21), we get

$$\begin{aligned} \rho_n(\varphi_0,\widehat{\varphi}_0) &= \mathbb{e}_{\overline{\alpha}\overline{h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha h}(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_{\beta}(\varphi_0)\Big)^{n-1} \cdot e^{-\|h\|^2} \cdot \int F(d\varphi_n)\mathbb{e}_{|h|^2}(\varphi_n) \\ &\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2}(\varphi_{n-1})\mathbb{e}_{|\beta|^4}(\varphi_n-\varphi_{n-1}) \sum_{\varphi_{n-2}\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \mathbb{e}_{|\beta|^2}(\varphi_n-\varphi_{n-2}) \\ &\cdot \sum_{\varphi_{n-3}\subseteq\varphi_{n-2}} \dots \sum_{\varphi_3\subseteq\varphi_4} \binom{\varphi_4}{\varphi_3} \mathbb{e}_{|\beta|^6}(\varphi_n-\varphi_3)\mathbb{e}_{1+|\beta|^2-|b|^4}(\varphi_3). \end{aligned} \tag{4.1.22}$$

Applying the same procedure repeatedly, we finally have

$$
\begin{aligned}
\rho_n(\varphi_0, \widehat{\varphi}_0) &= \mathbb{e}_{\overline{\alpha h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha h}(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_{\beta}(\varphi_0)\Big)^{n-1} \cdot e^{-\|h\|^2} \cdot \int F(d\varphi_n)\mathbb{e}_{|h|^2}(\varphi_n) \\
&\qquad \cdot \mathbb{e}_{|\alpha|^2(1+|\beta|^2+\ldots+|\beta|^{2(n-2)})+|\beta|^{2n}}(\varphi_n) \\
&= \mathbb{e}_{\overline{\alpha h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha h}(\varphi_0)\Big(\mathbb{e}_{\overline{\beta}}(\widehat{\varphi}_0)\mathbb{e}_{\beta}(\varphi_0)\Big)^{n-1} \cdot e^{-\|h\|^2} \cdot \int F(d\varphi_n)\mathbb{e}_{|h|^2}(\varphi_n)\mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}(\varphi_n) \\
&= e^{-\|h\|^2}\mathbb{e}_{\overline{\alpha\beta^{n-1}h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha\beta^{n-1}h}(\varphi_0) \cdot \int F(d\varphi_n)\mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})|h|^2}(\varphi_n) \\
&= e^{-\|h\|^2}\mathbb{e}_{\overline{\alpha\beta^{n-1}h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha\beta^{n-1}h}(\varphi_0) \cdot \|\mathbb{e}_{\sqrt{1-|\alpha|^2|\beta|^{2(n-1)}}h}\|^2 \\
&= e^{-\|h\|^2}\mathbb{e}_{\overline{\alpha\beta^{n-1}h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha\beta^{n-1}h}(\varphi_0) \cdot e^{(1-|\alpha|^2|\beta|^{2(n-1)})\|h\|^2} \\
&= e^{-|\alpha|^2|\beta|^{2(n-1)}\|h\|^2}\mathbb{e}_{\overline{\alpha\beta^{n-1}h}}(\widehat{\varphi}_0)\mathbb{e}_{\alpha\beta^{n-1}h}(\varphi_0) \\
&= e^{-|\alpha|^2|\beta|^{2(n-1)}\|h\|^2}\overline{\mathbb{e}_{\alpha\beta^{n-1}h}(\widehat{\varphi}_0)}\mathbb{e}_{\alpha\beta^{n-1}h}(\varphi_0).
\end{aligned}
$$

Hence, ω_n is a coherent state satifying (4.1.14).
From Proposition 3.12 and the definition of τ we know that

$$
\rho_{n]}(\underline{\varphi}^0, \underline{\varphi}^1) = \int F(d\varphi)\overline{g_n(\underline{\varphi}^1, \varphi)}g_n(\underline{\varphi}^0, \varphi)e^{-\|h\|^2}\mathbb{e}_h(\varphi + \sum_{l=1}^{n}\varphi_l^0)\mathbb{e}_{\overline{h}}(\varphi + \sum_{l=1}^{n}\varphi_l^1) \tag{4.1.23}
$$

with g_n defined by (3.1.15). If we insert $g = \mathbb{e}_\alpha \otimes \mathbb{e}_\beta$ into this formula we get

$$
\overline{g_n(\underline{\varphi}^1, \varphi)} = \prod_{i=1}^{n}\mathbb{e}_{\overline{\alpha}}(\varphi_i^1)\mathbb{e}_{\overline{\beta}}(\varphi_{i+1}^1 + \ldots + \varphi_n^1 + \varphi) = \mathbb{e}_{\overline{\beta}^n}(\varphi) \cdot (\mathbb{e}_{\overline{h}_1} \otimes \ldots \otimes \mathbb{e}_{\overline{h}_n})(\underline{\varphi}^1) \tag{4.1.24}
$$

and

$$
g_n(\underline{\varphi}^0, \varphi) = \prod_{i=1}^{n}\mathbb{e}_{\alpha}(\varphi_i^0)\mathbb{e}_{\beta}(\varphi_{i+1}^0 + \ldots + \varphi_n^0 + \varphi) = \mathbb{e}_{\beta^n}(\varphi) \cdot (\mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n})(\underline{\varphi}^0) \tag{4.1.25}
$$

with $h_i = \alpha\beta^{i-1}$. Thus we can continue (4.1.23) by

$$
\begin{aligned}
&\rho_{n]}(\underline{\varphi}^0, \underline{\varphi}^1) \\
&\quad = e^{-\|h\|^2}(\mathbb{e}_{\overline{h}_1} \otimes \ldots \otimes \mathbb{e}_{\overline{h}_n})(\underline{\varphi}^1)(\mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n})(\underline{\varphi}^0) \cdot \int F(d\varphi)\mathbb{e}_{|h|^2}(\varphi)\mathbb{e}_{|\overline{\beta}|^{2n}}(\varphi) \\
&\quad = e^{-\|h\|^2} \cdot (\mathbb{e}_{\overline{h}_1} \otimes \ldots \otimes \mathbb{e}_{\overline{h}_n})(\underline{\varphi}^1)(\mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n})(\underline{\varphi}^0) \cdot e^{|\beta|^{2n}\|h\|^2} \\
&\quad = e^{-(1-|\beta|^{2n})\|h\|^2} \cdot (\mathbb{e}_{\overline{h}_1} \otimes \ldots \otimes \mathbb{e}_{\overline{h}_n})(\underline{\varphi}^1)(\mathbb{e}_{h_1} \otimes \ldots \otimes \mathbb{e}_{h_n})(\underline{\varphi}^0).
\end{aligned}
$$

Hence, $\omega_{n]}$ is given by (4.1.15). $\square$

4.2 The Chain of Position Distributions in the Independent Case

We consider again the independent splitting with $g(\varphi_1, \varphi_2) := \mathbb{e}_\alpha\ (\varphi_1) \cdot \mathbb{e}_\beta\ (\varphi_2)$ and $\alpha,\ \beta : G \longrightarrow \mathbb{C}$, $|\alpha(x)|^2 + |\beta(x)|^2 = 1$ for all $x \in G$.
To a large extent we follow [24], where these calculations were done for the special case $\alpha = \beta = \frac{1}{\sqrt{2}}$ and diffuse reference measure ν.

Proposition 4.6. *Let $g(\varphi_1, \varphi_2) := \mathbb{e}_\alpha\ (\varphi_1) \cdot \mathbb{e}_\beta\ (\varphi_2)$ with $\alpha,\ \beta : G \longrightarrow \mathbb{C}$ and $|\alpha(x)|^2 + |\beta(x)|^2 = 1$ for all $x \in G$. For $n \geq 1$ and $Y \in \mathfrak{M}$ holds*

$$Q_n(Y) = \int\limits_Y F(d\varphi_1) \int F(d\varphi_2) \kappa(\varphi_1 + \varphi_2) \mathbb{e}_{b_n}\ (\varphi_2) \mathbb{e}_{a_n}\ (\varphi_1) \tag{4.2.1}$$

with $a_n(x) := |\alpha(x)|^2 |\beta(x)|^{2(n-1)}$, $b_n(x) := 1 - a_n(x)$ for all $x \in G$ and κ defined by (3.3.1).

PROOF. Because of Remark 3.19 we can insert the special choice of g into (3.3.7)and get

$$\begin{aligned} Q_n(Y) = \int\limits_Y F(d\varphi) \int F(d\varphi_n) \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_2}{\varphi_1} \ldots \binom{\varphi_n}{\varphi_{n-1}} \Big| & \mathbb{e}_\alpha(\varphi) \mathbb{e}_\beta(\varphi_n - \varphi_{n-1}) \\ & \cdot \mathbb{e}_\alpha(\varphi_1) \mathbb{e}_\beta(\varphi + \varphi_n - \varphi_1) \mathbb{e}_\alpha(\varphi_2 - \varphi_1) \mathbb{e}_\beta(\varphi + \varphi_n - \varphi_2) \mathbb{e}_\alpha(\varphi_3 - \varphi_2) \\ & \cdot \mathbb{e}_\beta(\varphi + \varphi_n - \varphi_3) \ldots \mathbb{e}_\alpha(\varphi_{n-1} - \varphi_{n-2}) \mathbb{e}_\beta(\varphi + \varphi_n - \varphi_{n-1}) \Big|^2 \\ & \cdot \rho(\varphi + \varphi_n, \varphi + \varphi_n). \end{aligned} \tag{4.2.2}$$

Because for $\varphi_1 \subseteq \varphi_2 \subseteq \ldots \subseteq \varphi_{n-1}$ there holds

$$\mathbb{e}_\alpha(\varphi_1) \mathbb{e}_\alpha(\varphi_2 - \varphi_1) \mathbb{e}_\alpha(\varphi_3 - \varphi_2) \ldots \mathbb{e}_\alpha(\varphi_{n-1} - \varphi_{n-2}) = \mathbb{e}_\alpha(\varphi_{n-1}), \tag{4.2.3}$$

(4.2.2) can be continued by

$$\begin{aligned} Q_n(Y) &= \int\limits_Y F(d\varphi) \int F(d\varphi_n) \big| \mathbb{e}_\alpha(\varphi) (\mathbb{e}_\beta(\varphi))^{n-1} \big|^2 \sum_{\varphi_1 \subseteq \ldots \subseteq \varphi_n} \binom{\varphi_2}{\varphi_1} \ldots \binom{\varphi_n}{\varphi_{n-1}} \\ & \quad \cdot \Big| \mathbb{e}_\alpha(\varphi_{n-1}) \cdot \mathbb{e}_\beta(\varphi_n - \varphi_{n-1}) \mathbb{e}_\beta(\varphi_n - \varphi_1) \mathbb{e}_\beta(\varphi_n - \varphi_2) \ldots \mathbb{e}_\beta(\varphi_n - \varphi_{n-1}) \Big|^2 \\ & \quad \cdot \rho(\varphi + \varphi_n, \varphi + \varphi_n) \\ &= \int\limits_Y F(d\varphi) \int F(d\varphi_n) \big| \mathbb{e}_\alpha(\varphi) (\mathbb{e}_\beta(\varphi))^{n-1} \big|^2 \\ & \quad \cdot \sum_{\varphi_{n-1} \subseteq \varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \big| \mathbb{e}_\alpha(\varphi_{n-1}) \big|^2 \big| \mathbb{e}_\beta(\varphi_n - \varphi_{n-1}) \big|^4 \sum_{\varphi_{n-2} \subseteq \varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \big| \mathbb{e}_\beta(\varphi_n - \varphi_{n-2}) \big|^2 \end{aligned}$$

$$\cdot \sum_{\varphi_{n-3}\subseteq\varphi_{n-2}} \dots \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \big| \mathbb{e}_\beta(\varphi_n - \varphi_2)\big|^2 \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \big|\mathbb{e}_\beta(\varphi_n - \varphi_1)\big|^2 \rho(\varphi+\varphi_n, \varphi+\varphi_n). \quad (4.2.4)$$

From

$$\mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_1) = \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_2) \cdot \mathbb{e}_{|\beta|^2}(\varphi_2 - \varphi_1)$$

it follows

$$\sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_1) = \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_2) \mathbb{e}_{|\beta|^2}(\varphi_2 - \varphi_1)$$

$$= \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_2) \sum_{\varphi_1\subseteq\varphi_2} \binom{\varphi_2}{\varphi_1} \mathbb{e}_{|\beta|^2}(\varphi_2 - \varphi_1) = \mathbb{e}_{|\beta|^2}(\varphi_n - \varphi_2) \mathbb{e}_{1+|\beta|^2}(\varphi_2). \quad (4.2.5)$$

If we insert this into (4.2.4), we get

$$Q_n(Y) = \int_Y F(d\varphi) \int F(d\varphi_n) \big|\mathbb{e}_\alpha(\varphi)(\mathbb{e}_\beta(\varphi))^{n-1}\big|^2 \cdot$$

$$\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \big|\mathbb{e}_\alpha\big|^2(\varphi_{n-1}) \big|\mathbb{e}_\beta(\varphi_n - \varphi_{n-1})\big|^4 \sum_{\varphi_{n-2}\subseteq\varphi_{n-1}} \binom{\varphi_{n-1}}{\varphi_{n-2}} \big|\mathbb{e}_\beta(\varphi_n - \varphi_{n-2})\big|^2 \cdot$$

$$\cdot \sum_{\varphi_{n-3}\subseteq\varphi_{n-2}} \dots \sum_{\varphi_2\subseteq\varphi_3} \binom{\varphi_3}{\varphi_2} \big|\mathbb{e}_\beta\big|^4(\varphi_n - \varphi_2) \mathbb{e}_{1+|\beta|^2}(\varphi_2) \rho(\varphi+\varphi_n, \varphi+\varphi_n). \quad (4.2.6)$$

Applying the same procedure repeatedly, as we already did in the proof of Proposition 4.5, we finally have

$$Q_n(Y) = \int_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2}(\varphi) \mathbb{e}_{|\beta|^{2(n-1)}}(\varphi) \cdot$$

$$\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2}(\varphi_{n-1}) \mathbb{e}_{|\beta|^4}(\varphi_n - \varphi_{n-2}) \mathbb{e}_{|\beta|^{2(n-1)}}(\varphi_n - \varphi_{n-1}) \cdot$$

$$\cdot\, \mathbb{e}_{1+|\beta|^2+\ldots+|\beta|^{2(n-2)}}(\varphi_{n-1}) \cdot \rho(\varphi+\varphi_n, \varphi+\varphi_n)$$

$$= \int_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2}(\varphi) (\mathbb{e}_{|\beta|^{2(n-1)}}(\varphi) \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2}(\varphi_{n-1}) \cdot$$

$$\cdot\, \mathbb{e}_{|\beta|^{2n}}(\varphi_n - \varphi_{n-1}) \mathbb{e}_{|1+|\beta|^2+\ldots+|\beta|^{2(n-2)}}(\varphi_{n-1}) \rho(\varphi+\varphi_n, \varphi+\varphi_n)$$

$$= \int_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2}(\varphi) \mathbb{e}_{|\beta|^{2(n-1)}}(\varphi) \cdot$$

$$\cdot \sum_{\varphi_{n-1}\subseteq\varphi_n} \binom{\varphi_n}{\varphi_{n-1}} \mathbb{e}_{|\alpha|^2(1+\ldots+|\beta|^{2(n-2)})}(\varphi_{n-1}) \mathbb{e}_{|\beta|^{2n}}(\varphi_n - \varphi_{n-1}) \kappa(\varphi+\varphi_n)$$

$$= \int_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2|\beta|^{2(n-1)}}(\varphi) \mathbb{e}_{|\alpha|^2(1+|\beta|^2+\ldots+|\beta|^{2(n-2)})+|\beta|^{2n})}(\varphi_n) \kappa(\varphi+\varphi_n)$$

$$= \int_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2|\beta|^{2(n-1)}}(\varphi) \mathbb{e}_{|\alpha|^2 \frac{1-|\beta|^{2(n-1)}}{|\alpha|^2}+|\beta|^{2n}}(\varphi_n) \kappa(\varphi+\varphi_n)$$

$$= \int\limits_Y F(d\varphi) \int F(d\varphi_n) \mathbb{e}_{|\alpha|^2|\beta|^{2(n-1)}}(\varphi) \mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}(\varphi_n) \kappa(\varphi + \varphi_n)$$

$$= \int\limits_Y F(d\varphi_1) \int F(d\varphi_2) \kappa(\varphi_1 + \varphi_2) \mathbb{e}_{b_n}(\varphi_2) \mathbb{e}_{a_n}(\varphi_1).$$

□

We want to give further characterization of the position distribution at time n for independent splittings. For $\varphi_1, \varphi_2 \in M$ and $n \geq 1$ we set

$$h_n(\varphi_1, \varphi_2) := \mathbb{e}_{a_n}(\varphi_1) \cdot \mathbb{e}_{b_n}(\varphi_2) \tag{4.2.7}$$

with $a_n(x) := |\alpha(x)|^2 |\beta(x)|^{2(n-1)}$ and $b_n(x) := 1 - a_n(x)$ for all $x \in G$.
For $Y \in \mathfrak{M}$, $n \geq 1$, $\varphi \in M$ let

$$H_n(\varphi, Y) := \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_Y(\varphi_1) h_n(\varphi_1, \varphi - \varphi_1). \tag{4.2.8}$$

All H_n are stochastic kernels:
They are measurable with respect to the first component. For fixed $\varphi \in M$, $H_n(\varphi, \cdot)$ is a measure on $[M, \mathfrak{M}]$. Moreover, from Lemma 2.13 we get

$$\begin{aligned} \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} h_n(\varphi_1, \varphi - \varphi_1) &= \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \mathbb{e}_{a_n}(\varphi_1) \cdot \mathbb{e}_{b_n}(\varphi - \varphi_1) \\ &= \mathbb{e}_{a_n + b_n}(\varphi) = \mathbb{e}_1(\varphi) = 1^{|\varphi|} = 1. \end{aligned} \tag{4.2.9}$$

Hence, $H_n(\varphi, \cdot)$ is a probability measure on $[M, \mathfrak{M}]$.

Corollary 4.7. *For all $n \geq 1$ and $Y \in \mathfrak{M}$ there holds*

$$Q_n(Y) = \int Q_\tau(d\varphi) H_n(\varphi, Y) \tag{4.2.10}$$

with Q_τ being the position distribution of τ and H_n the stochastic kernel given by (4.2.8).

PROOF. Using Propositions 4.6 and 2.11 we obtain

$$Q_n(Y) = \int\limits_Y F(d\varphi_1) \int F(d\varphi_2) \kappa(\varphi_1 + \varphi_2) \mathbb{e}_{a_n}(\varphi_1) \cdot \mathbb{e}_{b_n}(\varphi_2)$$

$$= \int F(d\varphi_2) \int F(d\varphi_1) \chi_Y(\varphi_1) \kappa(\varphi_1 + \varphi_2) \mathbb{e}_{a_n}(\varphi_1) \cdot \mathbb{e}_{b_n}(\varphi_2)$$

$$= \int F(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \kappa(\varphi) \chi_Y(\varphi_1) \mathbb{e}_{b_n}(\varphi - \varphi_1) \mathbb{e}_{a_n}(\varphi_1)$$

$$= \int F(d\varphi) \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \kappa(\varphi) \chi_Y(\varphi_1) h_n(\varphi_1, \varphi - \varphi_1) = \int F(d\varphi) \kappa(\varphi) H_n(\varphi, Y)$$

$$= \int Q_\tau(d\varphi) H_n(\varphi, Y).$$

□

Remark 4.8. *From Corollary 4.7 and Proposition 3.21 we see that the definition* (4.2.8) *of H_n is compatible with the definition of H_n given in Lemma 3.20.*

For functions $g(\varphi_1, \varphi_2) = \mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\beta(\varphi_2)$ with constants α and β satisfying $|\alpha|^2 + |\beta|^2 = 1$ there hold the following propositions:

Proposition 4.9. *Let for φ_1, $\varphi_2 \in M$ the function g be defined by $g(\varphi_1, \varphi_2) := \mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\beta(\varphi_2)$ with constants α, $\beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. For all $n \geq 1$ and $Y \in \mathfrak{M}$ holds*

$$Q_n(Y) = \int_Y F(d\varphi_1) \int F(d\varphi_2)\kappa(\varphi_1 + \varphi_2)(1 - |\alpha|^2|\beta|^{2(n-1)})^{|\varphi_2|}(|\alpha|^2|\beta|^{2(n-1)})^{|\varphi_1|}. \tag{4.2.11}$$

PROOF. This follows immediately from Proposition 4.6. □

Proposition 4.10. *Let g be defined as in Proposition 4.9. For all $n \geq 1$, $B \in \mathfrak{B}$ and $k \in \mathbb{N}$ there holds*

$$Q_n(\{\varphi : \varphi(B) = k\}) = \int Q_\tau(d\varphi)\mathbb{B}(\varphi(B), p)(k) \tag{4.2.12}$$

with $p := |\alpha|^2|\beta|^{2(n-1)}$ and $\mathbb{B}(m, p)(k) = \binom{m}{k}p^k(1-p)^{m-k}$ being the binomial distribution.

PROOF. For $\varphi \in M$ and $B \in \mathfrak{G}$ we denote again by $\varphi_{|_B}$ the restriction of φ to $\mathfrak{G} \cap B$, i.e. $\varphi_{|_B}(.) = \varphi(. \cap B)$. From the definition (4.2.8) of H_n one gets for all $B \in \mathfrak{B}$, $k \in \mathbb{N}$ $n \geq 1$ and $\varphi \in M$

$$\begin{aligned}
&H_n(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_{\{\widetilde{\varphi}:\widetilde{\varphi}(B)=k\}}(\varphi_1) h_n(\varphi_1, \varphi - \varphi_1) \\
&= \sum_{\substack{\varphi_1 \subseteq \varphi_B \\ |\varphi_1| = k}} \sum_{\varphi_2 \subseteq \varphi_{B^C}} \binom{\varphi}{\varphi_1 + \varphi_2} h_n(\varphi_1 + \varphi_2, \varphi - \varphi_1 - \varphi_2) \\
&= \binom{\varphi(B)}{k} (|\alpha|^2|\beta|^{2(n-1)})^k (1 - |\alpha|^2|\beta|^{2(n-1)})^{\varphi(B)-k} \sum_{\varphi_2 \subseteq \varphi_{B^C}} \binom{\varphi_{B^C}}{\varphi_2} h_n(\varphi_2, \varphi_{B^C} - \varphi_2) \\
&= \binom{\varphi(B)}{k} (|\alpha|^2|\beta|^{2(n-1)})^k (1 - |\alpha|^2|\beta|^{2(n-1)})^{\varphi(B)-k} \cdot 1 \\
&= \mathbb{B}(\varphi(B), |\alpha|^2|\beta|^{2(n-1)})(k) \\
&= \mathbb{B}(\varphi(B), p)(k).
\end{aligned}$$

□

Now we want to calculate the position distribution $Q_n(\{\varphi : \varphi(B) = k\})$ in the case of splitting functions $g(\varphi_1, \varphi_2) = \mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\beta(\varphi_2)$ with measurable functions α and β being not necessarily constant. We will use the notation introduced in [32].

Let the stochastic kernels H_n be defined as in (4.2.8). For $x \in G$ and $Y \in \mathfrak{M}$ we identify x with δ_x and set

$$H_n(x, Y) := \delta_{\mathfrak{o}}(Y) \cdot h_n(\mathfrak{o}, \delta_x) + \delta_{\delta_x}(Y) \cdot h_n(\delta_x, \mathfrak{o}).$$

Using (4.2.7) this means

$$\begin{aligned} H_n(x, Y) &= \delta_{\mathfrak{o}}(Y) \cdot \mathbb{e}_{a_n}(\mathfrak{o}) \cdot \mathbb{e}_{1-a_n}(\delta_x) + \delta_{\delta_x}(Y) \cdot \mathbb{e}_{a_n}(\delta_x) \cdot \mathbb{e}_{1-a_n}(\mathfrak{o}) \\ &= \delta_{\mathfrak{o}}(Y) \cdot (1 - a_n(x)) + \delta_{\delta_x}(Y) \cdot a_n(x) \qquad (4.2.13) \\ &= \Big((1 - a_n(x))\delta_{\mathfrak{o}} + a_n(x)\delta_{\delta_x}\Big)(Y). \qquad (4.2.14) \end{aligned}$$

Obviously, the so defined H_n are stochastic kernels from $[G, \mathfrak{G}]$ to $[M, \mathfrak{M}]$.
With the help of these we may give a description of the stochastic kernels H_n defined in (4.2.8).

Lemma 4.11. *For $Y \in \mathfrak{M}$ and $\varphi \in M^f$*

$$H_n(\varphi, Y) = \Big[\mathop{*}_{x \in \varphi} (H_n(x, \cdot))^{*\varphi(\{x\})}\Big](Y).$$

PROOF. Because of $\varphi \in M^f$ we may assume $\varphi = \sum_{i=1}^{m} \delta_{x_i}$ with $m \in \mathbb{N}_0$. Using (4.2.14), (4.2.8) Remark 2.16 and Lemma 2.17 we get

$$\begin{aligned} &\Big[\mathop{*}_{x \in \varphi} (H_n(x, \cdot))^{*\varphi(\{x\})}\Big](Y) = \Big[\mathop{*}_{x \in \varphi} (a_n(x)\delta_{\delta_x} + (1 - a_n(x))\delta_{\mathfrak{o}})^{*\varphi(\{x\})}\Big](Y) \\ &= \Big[\mathop{*}_{x \in \varphi} \sum_{k=0}^{\varphi(\{x\})} \binom{\varphi(\{x\})}{k} a_n^k(x)(1 - a_n(x))^{\varphi(\{x\})-k} \delta_{k\delta_x}\Big](Y) \\ &= \Big[\sum_{k_1=0}^{\varphi(\{x_1\})} \sum_{k_2=0}^{\varphi(\{x_2\})} \cdots \sum_{k_m=0}^{\varphi(\{x_m\})} \binom{\varphi(\{x_1\})}{k_1}\binom{\varphi(\{x_2\})}{k_2} \cdots \binom{\varphi(\{x_m\})}{k_m} \\ &\qquad a_n^{k_1}(x_1)(1 - a_n(x_1))^{\varphi(\{x_1\})-k_1} \cdots a_n^{k_m}(x_m)(1 - a_n(x_m))^{\varphi(\{x_m\})-k_m} \\ &\qquad \delta_{k_1\delta_{x_1}} * \delta_{k_2\delta_{x_2}} \cdots * \delta_{k_m\delta_{x_m}}\Big](Y) \qquad (4.2.15) \\ &= \Big(\sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} (\prod_{x \in \widehat{\varphi}} a_n(x)^{\widehat{\varphi}(\{x\})})(\prod_{y \in \varphi - \widehat{\varphi}} (1 - a_n(y))^{(\varphi-\widehat{\varphi})(\{y\})}) \cdot \delta_{\widehat{\varphi}}\Big)(Y) \\ &= \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \mathbb{e}_{a_n}(\widehat{\varphi}) \mathbb{e}_{1-a_n}(\varphi - \widehat{\varphi}) \delta_{\widehat{\varphi}}(Y) \\ &= \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} h_n(\widehat{\varphi}, \varphi - \widehat{\varphi}) \chi_Y(\widehat{\varphi}) \\ &= H_n(\varphi, Y). \end{aligned}$$

$\square$

Lemma 4.12. *Let for $\varphi_1, \varphi_2 \in M$ the function g be defined by $g(\varphi_1, \varphi_2) := \mathbb{e}_\alpha(\varphi_1)\mathbb{e}_\beta(\varphi_2)$ with measurable functions α , $\beta : G \to \mathbb{C}$ satisfying $|\alpha(x)|^2 + |\beta(x)|^2 = 1$ for all $x \in G$. For $n \geq 1$ and $Y \in \mathfrak{M}$ there holds*

$$Q_n(Y) = \int Q_\tau(d\varphi)\left[\mathop{*}_{x\in\varphi}(H_n(x,\cdot))^{*\varphi(\{x\})}\right](Y), \tag{4.2.16}$$

with Q_τ being the position distribution of τ and H_n the stochastic kernels given by (4.2.14).

PROOF. This is an immediate consequence of Lemma 4.11. □

Proposition 4.13. *Let g be defined as in Lemma 4.12. For all $n \geq 1$, $B \in \mathfrak{B}$ and $k \in \mathbb{N}$ there holds*

$$Q_n(\{\varphi : \varphi(B) = k\}) = \int Q_\tau(d\varphi) \sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1|=k}} \binom{\varphi_{|B}}{\varphi_1} \mathbb{e}_{a_n}(\varphi_1)\mathbb{e}_{1-a_n}(\varphi_{|B} - \varphi_1) \tag{4.2.17}$$

with $a_n(x) := |\alpha(x)|^2|\beta(x)|^{2(n-1)}$ for $x \in G$ and $\varphi_{|B}$ denoting the restriction of $\varphi \in M$ to $\mathfrak{G} \cap B$, i.e. $\varphi_{|B}(.) = \varphi(. \cap B)$.

PROOF. From the definition (4.2.8) of H_n one gets for all $B \in \mathfrak{B}$, $k \in \mathbb{N}$ $n \geq 1$ and $\varphi \in M^f$ using Lemma 2.8

$$H_n(\varphi, \{\widetilde{\varphi} : \widetilde{\varphi}(B) = k\}) = \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1} \chi_{\{\widetilde{\varphi}:\widetilde{\varphi}(B)=k\}}(\varphi_1) h_n(\varphi_1, \varphi - \varphi_1)$$

$$= \sum_{\varphi_1 \subseteq \varphi} \binom{\varphi}{\varphi_1 + \varphi_2} \chi_{\{\widetilde{\varphi}:\widetilde{\varphi}(B)=k\}}(\varphi_1) (\prod_{x\in\widehat{\varphi}} a_n(x)^{\widehat{\varphi}(\{x\})})(\prod_{y\in\varphi-\widehat{\varphi}} (1 - a_n(y))^{(\varphi-\widehat{\varphi})(\{y\})})$$

$$= \sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1|=k}} \sum_{\varphi_2 \subseteq \varphi_{|B^c}} \binom{\varphi}{\varphi_1 + \varphi_2} (\prod_{x\in\varphi_1+\varphi_2} a_n(x)^{(\varphi_1+\varphi_2)(\{x\})})(\prod_{y\in\varphi-(\varphi_1+\varphi_2)} (1 - a_n(y))^{(\varphi-(\varphi_1+\varphi_2))(\{y\})})$$

$$= \sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1|=k}} \sum_{\varphi_2 \subseteq \varphi_{|B^c}} \binom{\varphi_{|B}}{\varphi_1}\binom{\varphi_{|B^c}}{\varphi_2} (\prod_{x\in\varphi_1} a_n(x)^{\varphi_1(\{x\})})(\prod_{x\in\varphi_2} a_n(x)^{\varphi_2(\{x\})}) \cdot$$
$$(\prod_{y\in\varphi_{|B}-\varphi_1} (1 - a_n(y))^{(\varphi_{|B}-\varphi_1)(\{y\})})(\prod_{y\in\varphi_{|B^c}-\varphi_2} (1 - a_n(y))^{(\varphi_{|B^c}-\varphi_2)(\{y\})})$$

$$= \left(\sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1|=k}} \binom{\varphi_{|B}}{\varphi_1} (\prod_{x\in\varphi_1} a_n(x)^{\varphi_1(\{x\})})(\prod_{x\in\varphi_2} a_n(x)^{\varphi_2(\{x\})})\right) \cdot$$
$$\left(\sum_{\varphi_2 \subseteq \varphi_{|B^c}} \binom{\varphi_{|B^c}}{\varphi_2} (\prod_{y\in\varphi_{|B}-\varphi_1} (1 - a_n(y))^{(\varphi_{|B}-\varphi_1)(\{y\})})(\prod_{y\in\varphi_{|B^c}-\varphi_2} (1 - a_n(y))^{(\varphi_{|B^c}-\varphi_2)(\{y\})})\right)$$

$$= \left(\sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1|=k}} \binom{\varphi_{|B}}{\varphi_1} \mathbb{e}_{a_n}(\varphi_1)\mathbb{e}_{1-a_n}(\varphi_{|B} - \varphi_1)\right)\left(\sum_{\varphi_2 \subseteq \varphi_{|B^c}} \binom{\varphi_{|B^c}}{\varphi_2} \mathbb{e}_{a_n}(\varphi_2)\mathbb{e}_{1-a_n}(\varphi_{|B^c} - \varphi_2)\right)$$

$$= \left(\sum_{\substack{\varphi_1 \subseteq \varphi_{|B} \\ |\varphi_1| = k}} \binom{\varphi_{|B}}{\varphi_1} \mathbb{e}_{a_n}(\varphi_1) \mathbb{e}_{1-a_n}(\varphi_{|B} - \varphi_1) \right) \cdot 1.$$

□

Remark 4.14. *Let τ be a coherent initial state on $\mathcal{A}$ corresponding to a function $h \in \mathcal{L}^2(G, \nu)$. Then the position distribution Q_τ of τ defined in Remark 3.15 is a Poisson point process with intensity measure*

$$\Lambda(B) = \int_B \nu(dx)|h(x)|^2 \qquad (B \in \mathfrak{G}). \tag{4.2.18}$$

For details see [16] , [17].

Proposition 4.15. *Let τ be a coherent state on $\mathcal{A}$ corresponding to a function $h \in \mathcal{L}^2(G, \nu)$. Then for all $n \in \mathbb{N}$ the position distribution Q_n of the quantum Markov chain associated to τ and $\mathcal{E}_{\alpha,\beta}$ is a Poisson point process with intensity measure*

$$\Lambda_n(B) = \int_B \nu(dx)|(\alpha\beta^{n-1}h)(x)|^2 \qquad (B \in \mathfrak{G}). \tag{4.2.19}$$

PROOF. This follows immediately from Proposition 4.5 and Remark 4.14.

□

4.3 Invariance of Normal States

In section 3.6 of [26] there was shown that the vacuum state is the only normal state on the whole Fock space algebra that is invariant with respect to the independent beam splitting. For better comprehension we will cite the results obtained there. Since we are also interested in the measurement process we will give some conclusions on the evolution of the measurement apparatus.
For the description of the evolutions we use the notations of chapter 1.2. Note that we have $\mathcal{A} = \mathcal{B} = \mathfrak{L}(\mathcal{M})$ with the same type of measurement process in each step. But we will still use the notations $\mathcal{A}$ and $\mathcal{B}$ in some cases to distinguish between the observables of the quantum system and the measurement apparatus.
The mapping $\Lambda_Q^n : \mathcal{A} \longrightarrow \mathcal{A}$ defined for $A \in \mathcal{A}$ by

$$\Lambda_Q^n(A) := \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \ldots \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes A)) \ldots) \tag{4.3.1}$$

describes the evolution of the quantum system until time n.
The evolution of the measurement apparatus until time n is described by $\Lambda_M^n : \mathcal{B} \longrightarrow \mathcal{A}$ defined for $B \in \mathcal{B}$ by

$$\Lambda_M^n(B) = \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \ldots \mathcal{E}(\mathbb{1}_\mathcal{B} \otimes \mathcal{E}(B \otimes \mathbb{1}_\mathcal{A})) \ldots). \tag{4.3.2}$$

For an operator U from $\mathfrak{L}(\mathcal{M})$ we denote by $\tau_U : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ the mapping $\tau_U := U^*(\cdot)U$.

Definition 4.16. *Let $\alpha, \beta : G \longrightarrow \mathbb{C}$ be measurable functions with $|\alpha(x)| \leq 1$, $|\beta(x)| \leq 1$ for all $x \in G$, U_1, U_2 isometric operators from $\mathfrak{L}(\mathcal{M})$.*
We define $\Lambda_{M,\alpha,U_1} : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ and $\Lambda_{Q,\beta,U_2} : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ by

$$\begin{aligned} \Lambda_{M,\alpha,U_1} &:= \mathcal{E}_{\alpha,\sqrt{1-|\alpha|^2},U_1,\mathbb{1}}((\cdot) \otimes \mathbb{1}_{\mathcal{A}}) \\ &= \mathcal{V}^*_{\alpha,\sqrt{1-|\alpha|^2}}(U_1^*(\cdot)U_1 \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}_{\alpha,\sqrt{1-|\alpha|^2}} \\ &= \Lambda_{M,\alpha} \circ \tau_{U_1} \end{aligned} \tag{4.3.3}$$

with $\Lambda_{M,\alpha} := \Lambda_{M,\alpha,\mathbb{1}_{\mathcal{B}}} = \mathcal{E}_{\alpha,\sqrt{1-|\alpha|^2}}((\cdot) \otimes \mathbb{1}_{\mathcal{A}}) = \mathcal{V}^*_{\alpha,\sqrt{1-|\alpha|^2}}((\cdot) \otimes \mathbb{1}_{\mathcal{A}})\mathcal{V}_{\alpha,\sqrt{1-|\alpha|^2}}$ and

$$\begin{aligned} \Lambda_{Q,\beta,U_2} &:= \mathcal{E}_{\sqrt{1-|\beta|^2},\beta,\mathbb{1},U_2}(\mathbb{1}_{\mathcal{B}} \otimes (\cdot)) \\ &= \mathcal{V}^*_{\sqrt{1-|\beta|^2},\beta}(\mathbb{1}_{\mathcal{B}} \otimes U_2^*(\cdot)U_2)\mathcal{V}_{\sqrt{1-|\beta|^2},\beta} \\ &= \Lambda_{Q,\beta} \circ \tau_{U_2} \end{aligned} \tag{4.3.4}$$

with $\Lambda_{Q,\beta} := \Lambda_{Q,\beta,\mathbb{1}_{\mathcal{A}}} = \mathcal{E}_{\sqrt{1-|\beta|^2},\beta,\mathbb{1}}(\mathbb{1}_{\mathcal{B}} \otimes (\cdot)) = \mathcal{V}^*_{\sqrt{1-|\beta|^2},\beta}(\mathbb{1}_{\mathcal{B}} \otimes (\cdot))\mathcal{V}_{\sqrt{1-|\beta|^2},\beta}$ and the transition expectation $\mathcal{E}_{\alpha,\beta,U_1,U_2}$ defined by (3.1.6)*.*

Definition 4.17. *Let $\alpha, \beta : G \longrightarrow \mathbb{C}$ be measurable functions with $|\alpha(x)| \leq 1$, $|\beta(x)| \leq 1$ for all $x \in G$, U_1, U_2 isometric operators from $\mathfrak{L}(\mathcal{M})$.*
We define $\Lambda^n_{M,\alpha,U_1} : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ and $\Lambda^n_{Q,\beta,U_2} : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ by

$$\begin{aligned} \Lambda^1_{M,\alpha,U_1} &:= \Lambda_{M,\alpha,U_1} \\ \Lambda^n_{M,\alpha,U_1} &:= \Lambda^1_{M,\alpha,U_1} \circ \Lambda^{n-1}_{M,\alpha,U_1} \qquad (n \geq 2) \end{aligned} \tag{4.3.5}$$

and

$$\begin{aligned} \Lambda^1_{Q,\beta,U_2} &:= \Lambda_{Q,\beta,U_2} \\ \Lambda^n_{Q,\beta,U_2} &:= \Lambda^1_{Q,\beta,U_2} \circ \Lambda^{n-1}_{Q,\beta,U_2} \qquad (n \geq 2). \end{aligned} \tag{4.3.6}$$

We define $\Lambda^n_{M,\alpha,U_1,U_2} : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ by

$$\begin{aligned} \Lambda^1_{M,\alpha,U_1,U_2} &:= \Lambda_{M,\alpha,U_1} \\ \Lambda^n_{M,\alpha,U_1,U_2} &:= \Lambda^{n-1}_{Q,\sqrt{1-|\alpha|^2},U_2} \circ \Lambda^1_{M,\alpha,U_1} \qquad (n \geq 2). \end{aligned} \tag{4.3.7}$$

In [19] there were considered system evolutions of type (4.3.4). We want to recall some results.
By $\mathbf{Pr}_\Psi$ we denote the projection on Ψ :

$$\mathbf{Pr}_\Psi \widetilde{\Psi} := \langle \Psi, \widetilde{\Psi} \rangle \Psi. \tag{4.3.8}$$

For g_1, $g_2 \in \mathcal{L}^2(G, \nu)$ we denote by B_{g_1,g_2} the integral operator with kernel $\mathbb{e}_{g_2} \otimes \mathbb{e}_{\overline{g_1}}$, i.e. for $\Psi \in \mathcal{M}$ and $\varphi \in M$

$$B_{g_1,g_2}\Psi(\varphi) = \int F(d\widehat{\varphi})\mathbb{e}_{g_2}(\varphi) \cdot \mathbb{e}_{\overline{g_1}}(\widehat{\varphi})\Psi(\widehat{\varphi}). \tag{4.3.9}$$

We will use the operators of this type as test operators. The following Lemma is taken from [26] (Lemma 3.22).

Lemma 4.18. *Let L_1, $L_2 : \mathfrak{L}(\mathcal{M}) \longrightarrow \mathfrak{L}(\mathcal{M})$ be linear mappings being continuous with respect to the σ-weak and the uniform topology. Then it holds $L_1 = L_2$ if and only if*

$$L_1(B_{g_1,g_2}) = L_2(B_{g_1,g_2}) \qquad (g_1, g_2 \in \mathcal{L}^2(G,\nu)). \tag{4.3.10}$$

In [26] we also find the following formulae.

Lemma 4.19. *Using the notations of Definition 4.16 with $U_2 = \Gamma(\mathbf{v})$ where $\mathbf{v}$ is an isometry and B_{g_1,g_2} defined in (4.3.9) we have on exponential vectors from $\mathcal{M}$*

$$\Lambda_{Q,\beta}(B_{g_1,g_2})\mathbb{e}_h = e^{\langle g_1,\beta h\rangle}\mathbb{e}_{(1-|\beta|^2)h+\overline{\beta}g_2} \tag{4.3.11}$$

and

$$\Lambda_{Q,\beta,\Gamma(\mathbf{v})}(B_{g_1,g_2})\mathbb{e}_h = e^{\langle g_1,\mathbf{v}(\beta h)\rangle}\mathbb{e}_{(1-|\beta|^2)h+\overline{\beta}\mathbf{v}^* g_2}. \tag{4.3.12}$$

The following statements can be found in [26] (Prop. 3.28 and Lemma 3.33).
Let $\mathcal{U}_\beta$ denote the set of all operators $U \in \mathfrak{L}(\mathcal{M})$ such that τ_U commutes with $\Lambda_{Q,\beta}$.

Lemma 4.20. *Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$, $\beta : G \longrightarrow \mathbb{C}$ be a measurable function with $|\beta(x)| \leq 1$ for all $x \in G$, $U \in \mathcal{U}_\beta$ and $n \in \mathbb{N}$. Then*

(a) $\Lambda^n_{Q,\beta,U} = \Lambda_{Q,\beta^n,U^n}$,

(b) $\omega \circ \Lambda_{Q,\beta^n}(\mathbf{Pr}_{\mathbb{e}_0}) = \omega(O_{\mathbb{e}_{1-|\beta|^{2n}}})$.

Using these results we want to characterize Λ^n_{M,α,U_1}. For this we need some preparations.

Lemma 4.21. *Let $n \in \mathbb{N}$, $\alpha : G \longrightarrow \mathbb{C}$ a measurable function with $|\alpha(x)| \leq 1$, $\beta(x) := \sqrt{1-|\alpha(x)|^2}$ $(x \in G)$. Furthermore let U_1, U_2 be isometric operators from $\mathfrak{L}(\mathcal{M})$, $U_2 \in \mathcal{U}_\beta$. Then*

$$\Lambda^n_{M,\alpha,U_1,U_2} = \Lambda_{Q,\beta^{n-1},U_2^{n-1}} \circ \Lambda_{M,\alpha,U_1}.$$

PROOF. This follows immediately from the definitions of the considered mappings and Lemma 4.20. □

The next Lemma can be found as Proposition 3.30 in [26].
By $\emptyset^0$ we will denote the vacuum state, i.e.

$$\emptyset^0(A) := \langle \chi_{\{\mathfrak{o}\}}, A\chi_{\{\mathfrak{o}\}}\rangle = \langle \mathbb{e}_0, A\mathbb{e}_0\rangle, \quad (A \in \mathfrak{L}(\mathcal{M})), \tag{4.3.13}$$

where $\mathfrak{o}$ denotes the empty configuration in M.
If we want to stress the difference between quantum system and measurement apparatus we use $\emptyset^0_{\mathcal{A}}$ for the vacuum state on $\mathcal{A}$ and $\emptyset^0_{\mathcal{B}}$ for the vacuum state on $\mathcal{B}$.

Lemma 4.22. *Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of normal states on $\mathcal{M}$. If the sequence $(Q_n)_{n\in\mathbb{N}}$ of the corresponding position distributions defined in Definition 3.17 satisfies the condition*

$$\lim_{n\to\infty} Q_n(\{\mathfrak{o}\}) = 1$$

then

$$\lim_{n\to\infty} \omega_n = \emptyset$$

with respect to the $*$-weak and the norm topology.

Lemma 4.23. *(Corollary 3.32 in [26]) There is only one normal state ω with $Q_\omega(\{\mathfrak{o}\}) = 1$, namely $\omega = \emptyset$.*

Lemma 4.24. *For all normal states ω on $\mathcal{M}$ and measurable functions $\alpha : G \longrightarrow \mathbb{C}$ with $|\alpha(x)| \leq 1$ for all $x \in G$ it holds*

$$\omega \circ \Lambda_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0}) = \omega(O_{\mathbb{e}_{1-|\alpha|^2}}).$$

PROOF. Let $\beta := \sqrt{1-|\alpha|^2}$.
From (2.3.5) and (4.3.8) we know that $\mathbf{Pr}_{\mathbb{e}_0} = \langle \mathbb{e}_0, \cdot \rangle \mathbb{e}_0 = \Gamma(0)$. This implies

$$\begin{aligned}\Lambda_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0})\mathbb{e}_h &= \Lambda_{M,\alpha}(\Gamma(0))\mathbb{e}_h = \mathcal{V}^*_{\alpha,\beta}(\Gamma(0)\otimes \mathbb{1})\mathcal{V}_{\alpha,\beta}\mathbb{e}_h = \mathcal{V}^*_{\alpha,\beta}(\mathbb{e}_0 \otimes \mathbb{e}_{\beta h}) = \mathbb{e}_{|\beta|^2 h}\\ &= \mathbb{e}_{(1-|\alpha|^2)h} = O_{\mathbb{e}_{(1-|\alpha|^2)}}\mathbb{e}_h.\end{aligned}$$

□

Lemma 4.25. *For all normal states ω on $\mathcal{M}$ and measurable functions $\alpha : G \longrightarrow \mathbb{C}$ s. th. $|\alpha(x)| \leq 1$ for all $x \in G$, $n \in \mathbb{N}_0$ there holds*

$$\omega \circ (\Lambda_{Q,\beta^n,\mathbb{1}_\mathcal{B}} \circ \Lambda_{M,\alpha,\mathbb{1}_\mathcal{A}})(\mathbf{Pr}_{\mathbb{e}_0}) = \omega(O_{\mathbb{e}_{1-|\alpha|^2|\beta|^{2n}}})$$

with $\beta := \sqrt{1-|\alpha|^2}$.

PROOF. Let $\beta_1 := \sqrt{1-|\alpha|^2}^{\,n}$ and $\alpha_1 := \sqrt{1-|\beta_1|^2}$.

$$\begin{aligned}(\Lambda_{Q,\beta^n,\mathbb{1}_\mathcal{B}} \circ \Lambda_{M,\alpha,\mathbb{1}_\mathcal{A}})(\Gamma(0))\mathbb{e}_h &= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{1}_\mathcal{B} \otimes \mathcal{V}^*_{\alpha,\beta}(\Gamma(0)\otimes \mathbb{1}_\mathcal{A})\mathcal{V}_{\alpha,\beta})\mathcal{V}_{\alpha_1,\beta_1}\mathbb{e}_h\\ &= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{1}_\mathcal{B} \otimes \mathcal{V}^*_{\alpha,\beta}(\Gamma(0)\otimes \mathbb{1}_\mathcal{A})\mathcal{V}_{\alpha,\beta})\mathbb{e}_{\alpha_1 h}\otimes \mathbb{e}_{\beta_1 h}\\ &= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \mathcal{V}^*_{\alpha,\beta}(\Gamma(0)\mathbb{e}_{\alpha\beta_1 h}\otimes \mathbb{e}_{\beta\beta_1 h}))\\ &= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h}\otimes \mathbb{e}_{0+\beta_1|\beta|^2 h}) = \mathbb{e}_{(|\alpha_1|^2 h + |\beta_1|^2|\beta|^2 h)} = \mathbb{e}_{(1-|\beta|^{2n})h+|\beta|^{2(n+1)}h}\\ &= \mathbb{e}_{1-|\beta|^{2n}(1-|\beta|^2)} = \mathbb{e}_{1-|\alpha|^2|\beta|^{2n}} = O_{\mathbb{e}_{1-|\alpha|^2|\beta|^{2n}}}\mathbb{e}_h.\end{aligned}$$

□

Corollary 4.26. *Let α, β as in Lemma 4.25. Then*

$$\emptyset^0_\mathcal{A} \circ (\Lambda_{Q,\beta^n,\mathbb{1}_\mathcal{B}} \circ \Lambda_{M,\alpha,\mathbb{1}_\mathcal{A}}) = \emptyset^0_\mathcal{B}.$$

PROOF.

$$\begin{aligned} \emptyset^0_{\mathcal{A}} \circ (\Lambda_{Q,\beta^n} \circ \Lambda_{M,\alpha})(\mathbf{Pr}_{\mathbb{e}_0}) &= \emptyset^0(O_{\mathbb{e}_{1-|\alpha|^2|\beta|^{2n}}}) \\ &= \mathbb{e}_{1-|\alpha|^2|\beta|^{2n}}(\mathfrak{o}) = 1. \end{aligned}$$

From this follows $Q_{(\Lambda_{Q,\beta^n} \circ \Lambda_{M,\alpha})}(\mathfrak{o}) = 1$ and hence with Lemma 4.23, $\emptyset^0_{\mathcal{A}} \circ (\Lambda_{Q,\beta^n} \circ \Lambda_{M,\alpha}) = \emptyset^0_{\mathcal{B}}$.

□

The vacuum state $\emptyset^0$ defined in (4.3.13) is the only normal state on the Fock space algebra that is invariant with respect to our beam splittings.

Proposition 4.27. *(Proposition 3.35 in [26]) Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$ and $\beta : G \longrightarrow \mathbb{C}$ a measurable function with $|\beta(x)| < 1$ for all $x \in G$.*
Then it holds $\omega \circ \Lambda_{Q,\beta} = \omega$ if and only if $\omega = \emptyset^0$.

Repeated independent beam splittings always lead to the vacuum state (of course in both outcomes of the splitting, i.e. the transmitted and the reflected beam). We consider the reflected beam, i.e. in our interpretation, the measurement apparatus.

Proposition 4.28. *Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$ and $\alpha : G \longrightarrow \mathbb{C}$ a measurable function with $|\alpha(x)| < 1$ for all $x \in G$, $\beta := \sqrt{1-|\alpha|^2}$. Then*

$$\omega \circ \Lambda^n_{M,\alpha} \overset{weak-*,\|\cdot\|}{\underset{n\to\infty}{\longrightarrow}} \emptyset^0_{\mathcal{B}}. \tag{4.3.14}$$

PROOF. From Lemma 4.21 and Lemma 4.25 we conclude that for all $n \in \mathbb{N}$

$$\begin{aligned} \omega \circ \Lambda^n_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0}) &= \omega \circ (\Lambda^{n-1}_{Q,\beta^{n-1}} \circ \Lambda_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0})) \\ &= \omega(O_{\mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}}) = \int Q_\omega(d\varphi)\mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}(\varphi). \end{aligned}$$

Because of the assumption $|\alpha(x)| < 1$ we get $1 - |\alpha|^2|\beta|^{2(n-1)} \leq 1$ and hence for all $\varphi \in M$ and all $n \in \mathbb{N}$

$$\mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}(\varphi) \leq 1.$$

This implies

$$\lim_{n\to\infty} \mathbb{e}_{1-|\alpha|^2|\beta|^{2(n-1)}}(\varphi) = 1.$$

Consequently from the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \omega \circ \Lambda^n_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0}) = 1.$$

Lemma 4.24 and Lemma 4.22 complete the proof. □

Even if we add an evolution U of the quantum system, only the vacuum state is invariant (Proposition 3.37 in[26]).

Proposition 4.29. *Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$ and $\beta : G \longrightarrow \mathbb{C}$ a measurable function with $|\beta(x)| < 1$ for all $x \in G$.*
Furthermore, let $U \in \mathfrak{L}(\mathcal{M})$ be an unitary operator which leaves the vacuum invariant, i.e.

$$\tau_U(\mathbf{Pr}_{\mathbb{e}_0}) = U^*\mathbf{Pr}_{\mathbb{e}_0}U = \mathbf{Pr}_{\mathbb{e}_0}.$$

Then $\omega \circ \Lambda_{Q,\beta,U} = \omega$ if and only if $\omega = \emptyset^0$.

Consequently, Proposition 4.28 may be generalized in the following way.

Proposition 4.30. *Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$ and $\alpha : G \longrightarrow \mathbb{C}$ a measurable function with $|\alpha(x)| < 1$ for all $x \in G$.*
Furthermore, let U_1 be an unitary operator from $\mathfrak{L}(\mathcal{M})$ that leaves the vacuum invariant, i.e.

$$\tau_{U_1}(\mathbf{Pr}_{\mathbb{e}_0}) = U_1^*\mathbf{Pr}_{\mathbb{e}_0}U_1 = \mathbf{Pr}_{\mathbb{e}_0},$$

and $U_2 \in \mathcal{U}_{\sqrt{1-|\alpha|^2}}$ an operator of second quantization, i.e. $U_2 = \Gamma(\boldsymbol{v})$ with isometric $\boldsymbol{v} \in \mathfrak{L}(\mathcal{L}^2(G,\nu))$. Then

$$\omega \circ \Lambda^n_{M,\alpha,U_1,\Gamma(\boldsymbol{v})} \underset{n\to\infty}{\overset{weak-*,\|\cdot\|}{\longrightarrow}} \emptyset^0_{\mathcal{B}}. \tag{4.3.15}$$

PROOF. Let $\beta_1 := \sqrt{1-|\alpha|^2}$, $\alpha_1 := \sqrt{1-|\beta|^2}$ and $h \in \mathcal{L}^2(G,\nu)$. From $\tau_{U_1}(\mathbf{Pr}_{\mathbb{e}_0}) = \mathbf{Pr}_{\mathbb{e}_0}$ and Lemma 4.20 we conclude

$$\begin{aligned}
&\omega \circ \Lambda^n_{M,\alpha,U_1,U_2}(\mathbf{Pr}_{\mathbb{e}_0})\mathbb{e}_h = \omega \circ (\Lambda_{Q,\beta_1,U_2^{n-1}} \circ \Lambda_{M,\alpha,U_1}(\mathbf{Pr}_{\mathbb{e}_0}))\mathbb{e}_h \\
&= \omega \circ (\Lambda_{Q,\beta_1,U_2^{n-1}} \circ \Lambda_{M,\alpha}(U_1^*\mathbf{Pr}_{\mathbb{e}_0}U_1))\mathbb{e}_h = \omega \circ (\Lambda_{Q,\beta_1,U_2^{n-1}} \circ \Lambda_{M,\alpha}(\mathbf{Pr}_{\mathbb{e}_0}))\mathbb{e}_h \\
&= \omega \circ (\Lambda_{Q,\beta_1,U_2^{n-1}}(O_{\mathbb{e}_{1-|\alpha|^2}})\mathbb{e}_h = \omega \circ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{1} \otimes (U_2^{n-1})^* O_{\mathbb{e}_{|\beta|^2}} U_2^{n-1})\mathcal{V}_{\alpha_1,\beta_1}\mathbb{e}_h) \\
&= \omega \circ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{1} \otimes \Gamma((\boldsymbol{v}^{n-1})^*)O_{\mathbb{e}_{|\beta|^2}}\Gamma(\boldsymbol{v}^{n-1})\mathbb{e}_{|\beta_1|h}) \\
&= \omega \circ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \Gamma((\boldsymbol{v}^{n-1})^*)\mathbb{e}_{|\beta|^2(\boldsymbol{v}^{n-1}(|\beta_1|h))}) \\
&= \omega \circ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \mathbb{e}_{(\boldsymbol{v}^{n-1})^*(|\beta|^2(\boldsymbol{v}^{n-1}(|\beta_1|h)))}) \\
&= \omega \circ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \mathbb{e}_{|\beta|^2|\beta_1|h}) = \omega(O_{\mathbb{e}_{|\alpha_1|^2 h + |\beta_1|^2|\beta|^2 h}}) \\
&= \omega(O_{\mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})h}}).
\end{aligned}$$

The same conclusions as in the proof of Proposition 4.28 lead to the convergence to the vacuum state. $\square$

Chapter 5

Evolutions on the quasilocal Algebra

From section 4.3 we know that the vacuum state is the only normal state on the Fock space algebra that is invariant with respect to our beam splittings. This means that in a finite system a beam splitting (with $|\beta| < 1$) thins the configuration and leads for $n \to \infty$ to the vacuum state.
Now we want to consider the corresponding situation for infinite, locally finite systems (i.e. for locally normal states). We give a review of the results obtained in [26] and add some implications on the evolution of the measurement apparatus.
First we introduce the quasilocal algebra.

5.1 The Quasilocal Algebra on the Fock Space

Up to now we considered beam splittings on the symmetric Fock space $\mathcal{M}$ over some phase space G.
For the consideration of infinite quantum systems we need to pass over to the quasilocal algebra associated to $\mathcal{M}$.
Let $K \in \mathfrak{G}$ be a Borel set from G. By restricting everything to K we get

$$M_K := \{\varphi \in M : \varphi(K^c) = 0\} \ , \quad \mathfrak{M}_K := \mathfrak{M} \cap M_K \ , \quad F_K := F_{|\mathfrak{M}_K} \ ,$$

where $K^c := G \setminus K$ is the complement of K and $F_{|\mathfrak{M}_K}$ the restriction of the Fock space measure F to $\mathfrak{M}_K$.
Using this we obtain the Fock space $\mathcal{M}_K = \mathcal{L}^2(M_K, \mathfrak{M}_K, F_K)$ over K.
By setting $\Psi(\varphi) = 0$ for $\Psi \in \mathcal{M}_K$ and $\varphi \notin M_K$ we get $\mathcal{M}_K \subseteq \mathcal{M}$.
Now we can construct the local algebra over K. For more details see [17] and [19].
Let K be a Borel set from $\mathfrak{G}$. For $\varphi \in M$ let $\varphi_K := \varphi((\cdot) \cap K)$. Then there exists an unique isomorphism $\mathcal{I}_K : \mathcal{M}_K \otimes \mathcal{M}_{K^c} \longrightarrow \mathcal{M}$ which is characterized for F - almost all $\varphi \in M$ by

$$\mathcal{I}_K(\Psi_1 \otimes \Psi_2)(\varphi) = \Psi_1(\varphi_K) \cdot \Psi_2(\varphi_{K^c}) \qquad (\Psi_1 \in \mathcal{M}_K \ , \ \Psi_2 \in \mathcal{M}_{K^c}). \qquad (5.1.1)$$

Now let again $\mathfrak{B}$ be the ring of alle bounded Borel sets from $\mathfrak{G}$. For $K \in \mathfrak{B}$ $\mathbb{1}_{\mathcal{M}_K} = O_{M_K}$ denotes the identity in $\mathfrak{L}(\mathcal{M}_K)$. The algebra

$$\mathcal{C}_K := \mathcal{J}_K(\mathfrak{L}(\mathcal{M}_K))$$

with the embedding $\mathcal{J}_K : \mathfrak{L}(\mathcal{M}_K) \longrightarrow \mathfrak{L}(\mathcal{M})$ defined by

$$\mathcal{J}_K(A) := \mathcal{I}_K(A \otimes \mathbb{1}_{\mathcal{M}_{K^c}})\mathcal{I}_K^{-1} \qquad (A \in \mathfrak{L}(\mathcal{M}_K)\ ) \tag{5.1.2}$$

is a C^*-subalgebra of $\mathfrak{L}(\mathcal{M})$. $\mathcal{C}_K$ is called LOCAL ALGEBRA over K. These local algebras describe finite boson systems in K.
On exponential vectors this embedding can be described the following way.

Lemma 5.1. *([26], Lemma 4.1) For $K \in \mathfrak{B}$ and $A \in \mathfrak{L}(\mathcal{M}_K)$ we have on exponential vectors from $\mathcal{M}$*

$$\mathcal{J}_K(A) = \mathcal{S}^c(A \otimes O_{M_{K^c}})\mathcal{D}^c. \tag{5.1.3}$$

Now we can define the quasilocal algebra (see chapter 1.2).

$$\mathcal{C} := \overline{\bigcup_{K \in \mathfrak{B}} \mathcal{C}_K}$$

where the bar denotes the closure with respect to the uniform topology in $\mathfrak{L}(\mathcal{M})$, defines another C^*-subalgebra of $\mathfrak{L}(\mathcal{M})$.
The pair $[\mathcal{C}, (\mathcal{C}_K)_{K \in \mathfrak{B}}]$ is called QUASILOCAL ALGEBRA over G (in the sense of [8], Def. 2.6.3). $\mathcal{C}$ is used to describe infinite boson systems being locally finite (i.e. finite in bounded areas $K \in \mathfrak{B}$).
Now we consider position distributions for locally normal states. The following definition is a reformulation of Definition 1.9.

Definition 5.2. *Let ω be a state on the quasilocal algebra $\mathcal{C}$. ω is called LOCALLY NORMAL if it is a normal state on all local algebras $\mathcal{C}_K,\ K \in \mathfrak{B}$, i.e. for all $K \in \mathfrak{B}$ there exists a trace class operator ρ_K on $\mathcal{M}_K$ such that*

$$\omega(\mathcal{J}_K(A)) = Tr(\rho_K A) \qquad (A \in \mathfrak{L}(\mathcal{M}_K)).$$

The following proposition shows that it is possible to use the notion of position distribution also in this situation.

Proposition 5.3. *([17] , Theorem 2.15) Let ω be a locally normal state on $\mathcal{C}$. Then there exists exactly one point process Q_ω (the so-called position distribution) which satisfies for all $K \in \mathfrak{B}$ and all $Y \in \mathfrak{M}_K$*

$$\int Q_\omega(d\varphi)\chi_Y(\varphi_K) = \omega(\mathcal{J}_K(O_Y)).$$

If the locally normal state ω possesses an extension to a normal state on $\mathfrak{L}(\mathcal{M})$ then both point processes Q_ω coincide. In this sense both definitions are compatible. For more details see [17, 26].

Analogously to (5.1.1) for $K \in \mathfrak{B}$ there exists an isomorphism $\mathcal{I}_K^{(2)} : \mathcal{M}_K^{\otimes 2} \otimes \mathcal{M}_{K^c}^{\otimes 2} \longrightarrow \mathcal{M} \otimes \mathcal{M}$ given for $\Psi_1, \Psi_2 \in \mathcal{M}_K$, $\hat{\Psi}_1, \hat{\Psi}_2 \in \mathcal{M}_{K^c}$, $\varphi, \hat{\varphi} \in M$ by

$$\mathcal{I}_K^{(2)}(\Psi_1 \otimes \Psi_2 \otimes \hat{\Psi}_1 \otimes \hat{\Psi}_2)(\varphi, \hat{\varphi}) = \Psi_1(\varphi_K)\Psi_2(\hat{\varphi}_K)\hat{\Psi}_1(\varphi_{K^c})\hat{\Psi}_2(\hat{\varphi}_{K^c}),$$

or, equivalently, by

$$\mathcal{I}_K^{(2)}(\Psi_1 \otimes \Psi_2 \otimes \hat{\Psi}_1 \otimes \hat{\Psi}_2) = \Big(\mathcal{I}_K(\Psi_1 \otimes \hat{\Psi}_1)\Big) \otimes \Big(\mathcal{I}_K(\Psi_2 \otimes \hat{\Psi}_2)\Big).$$

The following Lemma shows that the operator $\mathcal{V}_{\alpha,\beta}$ from section 4.1 may be expressed by the tensor product of its restrictions to $\mathcal{M}_K$ and $\mathcal{M}_{K^c}$, respectively.

Lemma 5.4. *([26] , Lemma 4.4) Let α, $\beta : G \longrightarrow \mathbb{C}$ be measurable mappings satisfying* (4.1.1) *and let $\mathcal{V}_{\alpha,\beta} : \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{M}$ be defined by* (4.1.2)*. For arbitrary $K \in \mathfrak{B}$ we set $\mathcal{V}_{\alpha,\beta}^K := \mathcal{V}_{\alpha,\beta_{|\mathcal{M}_K}}$, i.e. $\mathcal{V}_{\alpha,\beta}^K$ is the restriction of $\mathcal{V}_{\alpha,\beta}$ to functions from $\mathcal{M}_K$. Then for all $\Psi \in \mathcal{M}$ we have $\mathcal{V}_{\alpha,\beta}^K\Psi \in \mathcal{M}_K \otimes \mathcal{M}_K$ and the following identity holds*

$$\mathcal{V}_{\alpha,\beta} = \mathcal{I}_K^{(2)}(\mathcal{V}_{\alpha,\beta}^K \otimes \mathcal{V}_{\alpha,\beta}^{K^c})\mathcal{I}_K^{-1}. \tag{5.1.4}$$

The following Lemma can also be found in [26]. Since $\mathcal{V}_{\alpha,\beta}^*$ looks a bit different in our situation, where multiple points are allowed, we will give the proof.

Lemma 5.5. *For arbitrary $K \in \mathfrak{B}$ we set $(\mathcal{V}_{\alpha,\beta}^*)^K := \mathcal{V}^*_{\alpha,\beta_{|\mathcal{M}_K \otimes \mathcal{M}_K}}$, i.e. $(\mathcal{V}_{\alpha,\beta}^*)^K$ is the restriction of $\mathcal{V}_{\alpha,\beta}^*$ to functions from $\mathcal{M}_K \otimes \mathcal{M}_K$.*
Then $(\mathcal{V}_{\alpha,\beta}^*)^K = (\mathcal{V}_{\alpha,\beta}^K)^*$ and

$$\mathcal{V}_{\alpha,\beta}^* = \mathcal{I}_K((\mathcal{V}_{\alpha,\beta}^K)^* \otimes (\mathcal{V}_{\alpha,\beta}^{K^c})^*)(\mathcal{I}_K^{(2)})^{-1}. \tag{5.1.5}$$

PROOF. Let $\Phi \in \mathcal{M} \otimes \mathcal{M}$, $K \in \mathfrak{B}$ and $\Psi \in \mathcal{M}$. Using Proposition 2.11 we get

$$\begin{aligned}
&\langle \Phi, \mathcal{V}_{\alpha,\beta}^K \Psi\rangle_{\mathcal{M}^2} \\
&= \int F(d\varphi_1) \int F(d\varphi_2)\overline{\Phi(\varphi_1,\varphi_2)}\mathrm{e}_\alpha(\varphi_1)\mathrm{e}_\beta(\varphi_2) \cdot \chi_{M_K \times M_K}(\varphi_1,\varphi_2) \cdot \Psi(\varphi_1+\varphi_2) \\
&= \int_{M_K} F(d\varphi) \sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \overline{\Phi(\widehat{\varphi}, \varphi - \widehat{\varphi})}\mathrm{e}_\alpha(\widehat{\varphi})\mathrm{e}_\beta(\varphi - \widehat{\varphi})\Psi(\varphi) \\
&= \int_{M_K} F(d\varphi)\Psi(\varphi)\overline{\sum_{\widehat{\varphi} \subseteq \varphi} \binom{\varphi}{\widehat{\varphi}} \Phi(\widehat{\varphi}, \varphi - \widehat{\varphi})\mathrm{e}_{\overline{\alpha}}(\widehat{\varphi})\mathrm{e}_{\overline{\beta}}(\varphi - \widehat{\varphi})} = \langle \mathcal{V}^*_{\alpha,\beta_{|\mathcal{M}_K \otimes \mathcal{M}_K}}\Phi, \Psi\rangle_{\mathcal{M}},
\end{aligned}$$

hence, $(\mathcal{V}^*_{\alpha,\beta})^K = (\mathcal{V}^K_{\alpha,\beta})^*$. Furthermore, we get from Lemma 2.9

$$\begin{aligned}\mathcal{V}^*_{\alpha,\beta} &= (\mathcal{I}^{(2)}_K(\mathcal{V}^K_{\alpha,\beta} \otimes \mathcal{V}^{K^c}_{\alpha,\beta})\mathcal{I}^{-1}_K)^* = (\mathcal{I}^{-1}_K)^*(\mathcal{V}^K_{\alpha,\beta})^* \otimes (\mathcal{V}^{K^c}_{\alpha,\beta})^*(\mathcal{I}^{(2)}_K)^* \\ &= \mathcal{I}_K((\mathcal{V}^K_{\alpha,\beta})^* \otimes (\mathcal{V}^{K^c}_{\alpha,\beta})^*)(\mathcal{I}^{(2)}_K)^{-1}.\end{aligned}$$

□

Lemma 5.6. *([26], Prop. 4.6) Let $\mathcal{E}_{\alpha,\beta}$ be given by (4.1.4). For all $K \in \mathfrak{B}$ there holds*

$$\mathcal{E}_{\alpha,\beta}(C) \in \mathcal{C}_K \qquad (C \in \mathcal{C}_K \otimes \mathcal{C}_K).$$

Especially, in the proof it was shown that for $A_1, A_2 \in \mathfrak{L}(\mathcal{M}_K)$

$$\mathcal{E}_{\alpha,\beta}(\mathcal{J}_K(A_1) \otimes \mathcal{J}_K(A_2)) = \mathcal{J}_K((\mathcal{V}^K_{\alpha,\beta})^*(A_1 \otimes A_2)\mathcal{V}^K_{\alpha,\beta}). \tag{5.1.6}$$

For the von Neumann algebra $\mathfrak{L}(\mathcal{M})$ we used the von Neumann tensor product $\mathfrak{L}(\mathcal{M}) \otimes \mathfrak{L}(\mathcal{M}) = \mathfrak{L}(\mathcal{M} \otimes \mathcal{M})$. In the case of the C^*-algebra we define a quasilocal algebra $\mathcal{C} \otimes_{ql} \mathcal{C}$ as C^*-algebra generated by the local algebras $\mathcal{C}_{K,K'}$ being the von Neumann algebra $\mathcal{C}_K \otimes \mathcal{C}_{K'}$.

Lemma 5.7. *([26], Corollary 4.7) For all $C \in \mathcal{C} \otimes_{ql} \mathcal{C}$ we have $\mathcal{E}_{\alpha,\beta}(C) \in \mathcal{C}$.*

Independent beam splitting is the quantum mechanical counterpart for the independent splitting of point processes ([26, 28]). From point process theory we know that if the two parts of a random configuration after the splitting are independent then the configuration before the splitting was distributed according to a Poisson process (Satz 1 in [21]). Theorem 3.4 of [28] contains the analogue result for the quantum case.
Let $\mathcal{L}^2_{loc}(G,\nu) := \{f : G \longrightarrow \mathbb{C} \ : \ f_K \in \mathcal{L}^2(G,\nu)\ \forall K \in \mathfrak{B}\}$, where for $f : G \longrightarrow \mathbb{C}$ $f_K := f \cdot \chi_K$ denotes the restriction of f to $K \in \mathfrak{G}$. Furthermore, let for $h \in \mathcal{L}^2_{loc}(G,\nu)$ Φ^h denote the coherent state corresponding to h.

Proposition 5.8. *([28] ,Theorem 3.4) Let ω be a locally normal state on $\mathcal{C}$ and α, $\beta \in \mathbb{C} \setminus \{0\}$ with $|\alpha|^2 + |\beta|^2 = 1$ fulfilling*

$$\omega(\mathcal{E}_{\alpha,\beta}(A \otimes B)) = \omega(\mathcal{E}_{\alpha,\beta}(A \otimes \mathbb{1})) \cdot \omega(\mathcal{E}_{\alpha,\beta}(\mathbb{1} \otimes B)) \quad (A, B \in \mathcal{C}). \tag{5.1.7}$$

Then there exists a function $h \in \mathcal{L}^2_{loc}(G,\nu)$ such that $\omega = \Phi^h$. Conversely, all coherent states Φ^h fulfil equation (5.1.7).

Now we want to answer the question under which conditions $\mathcal{E}_{\alpha,\beta,U_1,U_2}$ defined in (3.1.6) leaves the local and quasilocal, respectively, algebras invariant. For this we still need the first part of Proposition 4.8 from [26].
Remember that in our case

$$\begin{aligned}\mathcal{E}_{\alpha,\beta,U_1,U_2}(B \otimes A) &= \mathcal{V}^*_{\alpha,\beta}(U_1^*BU_1 \otimes U_2^*AU_2)\mathcal{V}_{\alpha,\beta} \\ \mathcal{E}_{\alpha,\beta,U_1,U_2} &= \mathcal{E}_{\alpha,\beta} \circ (\tau_{U_1} \otimes \tau_{U_2}),\end{aligned}$$

where we again used the notation

$$\tau_U(.) \quad = \quad U^*(.)U.$$

Lemma 5.9. *Assume the isometry $U \in \mathfrak{L}(\mathcal{M})$ is such that for each $K \in \mathfrak{B}$ there exists a set $K' \in \mathfrak{B}$ such that $\tau_U(A) \in \mathcal{C}_{K'}$ for all $A \in \mathcal{C}_K$. Then*

$$\tau_U\left(\bigcup_{K\in\mathfrak{B}} \mathcal{C}_K\right) \subseteq \bigcup_{K\in\mathfrak{B}} \mathcal{C}_K.$$

Now we can answer our question from above: The proof uses exactly the same arguments as the proof of Propostion 4.8 in [26].

Lemma 5.10. *Under the conditions of Lemma 5.9 there holds $\mathcal{E}_{\alpha,\beta,U_1,U_2}(\mathcal{C} \otimes_{ql} \mathcal{C}) \subseteq \mathcal{C}$. Moreover, for each locally normal state ω on $\mathcal{C}$ the state $\omega \circ \mathcal{E}_{\alpha,\beta,U_1,U_2}$ is a locally normal state on $\mathcal{C} \otimes_{ql} \mathcal{C}$.*

Corollary 5.11. *Let U_1, U_2 fulfill the conditions of Lemma 5.9 and let $\alpha : G \longrightarrow \mathbb{C}$ be a measurable function with $|\alpha(x)| < 1$ for all $x \in G$. Then for $\Lambda_{M,\alpha,U_1,U_2}$ defined in* (4.3.7) *it holds*

$$\Lambda_{M,\alpha,U_1,U_2}(\mathcal{C}) \subseteq \mathcal{C}$$

and $\Lambda^*_{M,\alpha,U_1,U_2}$ maps locally normal states into locally normal states.

We complete this section with a characterization of convergence in the weak-$*$ topology taken from [26].

Lemma 5.12. *([26], Lemma 4.12)*
Let $(\omega_n)_{n\in\mathbb{N}}$, ω be locally normal states. Then

$$\omega_n \underset{n\to\infty}{\overset{weak-*}{\longrightarrow}} \omega$$

iff for all $K \in \mathfrak{B}$ and all $g_1, g_2 \in \mathcal{L}^2(K)$

$$\omega_n(\mathcal{J}_K(B_{g_1,g_2})) \underset{n\to\infty}{\overset{weak-*}{\longrightarrow}} \omega(\mathcal{J}_K(B_{g_1,g_2})).$$

5.2 Invariance of Locally Normal States

The vacuum state is also the only *locally normal* state that is invariant under independent beam splittings. In this section we will cite the corresponding results from section 4.2 of [26] and add some implications on the evolution of the measurement apparatus.
The vacuum state will again be denoted by $\emptyset^0$.

From (5.1.6) we know that for all $K \in \mathfrak{B}$ and $A \in \mathfrak{L}(\mathcal{M}_K)$

$$\Lambda_{Q,\beta}(\mathcal{J}_K(A)) \quad = \quad \mathcal{J}_K(\Lambda_{Q,\beta_{|K}}(A)) \quad \text{and} \tag{5.2.1}$$
$$\Lambda_{M,\alpha}(\mathcal{J}_K(A)) \quad = \quad \mathcal{J}_K(\Lambda_{M,\alpha_{|K}}(A)). \tag{5.2.2}$$

For all locally normal states ω on $\mathcal{C}$ and $K \in \mathfrak{B}$ we hence have

$$\omega \circ \Lambda_{M,\alpha}(\mathcal{J}_K(A)) = \omega \circ \mathcal{J}_K(\Lambda_{M,\alpha_{|K}}(A)) \quad \text{and} \tag{5.2.3}$$
$$\omega \circ \Lambda_{Q,\beta}(\mathcal{J}_K(A)) = \omega \circ \mathcal{J}_K(\Lambda_{Q,\beta_{|K}}(A)). \tag{5.2.4}$$

The vacuum state is also the only locally normal state invariant under the beam splitting.

Proposition 5.13. *([26], Prop. 4.13) Let ω be a locally normal state on $\mathcal{C}$, $n \in \mathbb{N}$ and $\beta : G \longrightarrow \mathbb{C}$ a measurable function satisfying $|\beta(x)| < 1$ for all $x \in G$.*
Then $\omega \circ \Lambda^n_{Q,\beta} = \omega$ if and only if $\omega = \emptyset^0$.

From (5.1.6), (5.2.1) and (5.2.2) we conclude for all $K \in \mathfrak{B}$, $A \in \mathfrak{L}(\mathcal{M}_K)$ and $n \in \mathbb{N}$

$$\begin{aligned}
\Lambda^n_{M,\alpha}(\mathcal{J}_K(A)) &= (\Lambda^{n-1}_{Q,\sqrt{1-|\alpha|^2}} \circ \Lambda_{M,\alpha})(\mathcal{J}_K(A)) \\
&= \Lambda^{n-1}_{Q,\sqrt{1-|\alpha|^2}}(\mathcal{J}_K(\Lambda_{M,\alpha_{|K}}(A)) = \Lambda_{Q,\sqrt{1-|\alpha|^2}^{(n-1)}}(\mathcal{J}_K(\Lambda_{M,\alpha_{|K}}(A)) \\
&= \mathcal{J}_K(\Lambda_{Q,\sqrt{1-|\alpha|^2}^{(n-1)}_{|K}}(\Lambda_{M,\alpha_{|K}}(A)) \\
&= \mathcal{J}_K(\Lambda_{Q,\sqrt{1-|\alpha|^2}^{(n-1)}} \circ \Lambda_{M,\alpha})_{|K})(A).
\end{aligned}$$

So, we also have

$$\Lambda^n_{M,\alpha}(\mathcal{J}_K(A)) = \mathcal{J}_K(\Lambda^n_{M,\alpha_{|K}}(A)). \tag{5.2.5}$$

From (5.2.1) we also get

$$\Lambda^n_{Q,\beta}(\mathcal{J}_K(A)) = \mathcal{J}_K(\Lambda^n_{Q,\beta_{|K}}(A)). \tag{5.2.6}$$

Now we extend Proposition 4.28 to the case of locally normal states.

Proposition 5.14. *Let ω be a locally normal state on $\mathfrak{L}(\mathcal{M})$ and $\alpha : G \longrightarrow \mathbb{C}$ a measurable function with $|\alpha(x)| < 1$ for all $x \in G$, $\beta := \sqrt{1-|\alpha|^2}$. Then it holds*

$$\omega \circ \Lambda^n_{M,\alpha} \underset{n\to\infty}{\overset{weak-*,\|\cdot\|}{\longrightarrow}} \emptyset^0. \tag{5.2.7}$$

PROOF. From (5.2.5) we get for all $K \in \mathfrak{B}$ and $n \in \mathbb{N}$

$$\Lambda^n_{M,\alpha} \circ \mathcal{J}_K = \mathcal{J}_K \circ \Lambda^n_{M,\alpha_{|K}}. \tag{5.2.8}$$

So, from Proposition 4.28 it follows for all $K \in \mathfrak{B}$ that

$$(\omega \circ \Lambda^n_{M,\alpha})_K = \omega_K \circ \Lambda^n_{M,\alpha_{|K}} \underset{n\to\infty}{\overset{\sigma-weak,\|\cdot\|}{\longrightarrow}} \emptyset^0. \tag{5.2.9}$$

As the familiy $(\omega \circ \Lambda^n_{M,\alpha})_K$ is equicontinuous, $(\omega \circ \Lambda^n_{M,\alpha})_{n\in\mathbb{N}}$ does not only converge weak-$*$ly with respect to $\bigcup_{K\in\mathfrak{B}} \mathcal{C}_K$, but also with respect to the uniform closure of $\mathcal{C}$. This means that the convergence is usual weak-$*$ convergence. $\square$

Analogously to Proposition 5.14 we get the following Corollary (see also Proposition 4.14 in [26].

Corollary 5.15. *Let ω be a locally normal state on $\mathfrak{L}(\mathcal{M})$ and $\beta : G \longrightarrow \mathbb{C}$ a measurable function with $|\beta(x)| < 1$ for all $x \in G$, $\alpha := \sqrt{1 - |\beta|^2}$. Then*

$$\omega \circ \Lambda^n_{Q,\beta} \underset{n\to\infty}{\overset{weak-*,\|\cdot\|}{\longrightarrow}} \emptyset^0. \tag{5.2.10}$$

Chapter 6

Beam Splittings and Contractions

For normal and locally normal states ω the vacuum state is the only one that satisfies the invariance equation $\omega \circ \Lambda_{Q,\beta} = \omega$ ([26], chapters 3.6 and 4.2).
In the case of a normal state ω even the invariance equation $\omega \circ \Lambda_{Q,\beta,U} = \omega$ can be fulfilled only by the vacuum state ([26], Proposition 3.37).
But if we consider strictly locally normal states, i.e. locally normal ones that can not be extended to normal states on the whole Fock space $\mathcal{M}$, one can find states ω, that are invariant with respect to $\Lambda_{Q,\beta,U}$ for certain β and U.
In this chapter we will mainly deal with Λ^n_{Q,β,U_2} and $\Lambda^n_{M,\alpha,U_1,U_2}$ where the U_i are the second quantizations of contraction operators. Thereby we will give a review of the invariance results in chapter 5 of [26]. There it was shown that a contraction can "compensate" the loss caused by the splitting and thus lead to invariance.
In doing so we use a different representation of the contraction operator. Our second focus will be again the evolution of the measurement apparatus.

In the following we consider only a d-dimensional Euclidian phase space $G = \mathbb{R}^d$ equiped with the Lebesgue measure $\nu = \ell^d$.

6.1 Definition and Basic Properties of the Contraction Operator

For arbitrary $t \in \mathbb{R}$ we define an operator $\mathbf{v}_t \in \mathfrak{L}(\mathcal{L}^2(\mathbb{R}^d))$ by

$$(\mathbf{v}_t(h))(x) := e^{\frac{td}{2}} \cdot h(e^t x) \qquad (h \in \mathcal{L}^2(\mathbb{R}^d),\ x \in \mathbb{R}^d). \tag{6.1.1}$$

We will call this operator a contraction for arbitrary $t \in \mathbb{R}$. In the usual sense it is only a contraction for $t < 0$.

Lemma 6.1. *Let for $t \in \mathbb{R}$ the operator $\mathbf{v}_t$ be given by* (6.1.1)*. Then $\mathbf{v}_t$ is an unitary operator from $\mathfrak{L}(\mathcal{L}^2(\mathbb{R}^d))$ with $\mathbf{v}_t^* = \mathbf{v}_t^{-1} = \mathbf{v}_{-t}$.*

PROOF. Let $f,\, h \in \mathcal{L}^2(\mathbb{R}^d)$. Then

$$\langle f, \mathbf{v}_t h\rangle = \int\limits_{\mathbb{R}^d} \ell^d(dx)\overline{f(x)}e^{\frac{td}{2}}h(e^t x) = \int\limits_{\mathbb{R}^d} \ell^d(dy)\overline{f(e^{-t}y)}h(y)e^{-\frac{td}{2}} = \langle \mathbf{v}_{-t}f, h\rangle \quad (6.1.2)$$

and

$$\begin{aligned}\langle \mathbf{v}_t f, \mathbf{v}_t h\rangle &= \int\limits_{\mathbb{R}^d} \ell^d(dx)e^{\frac{td}{2}}\overline{f(e^t x)}e^{\frac{td}{2}}h(e^t x) = \int\limits_{\mathbb{R}^d} \ell^d(dx)e^{td}\overline{f(e^t x)}h(e^t x) \\ &= \int\limits_{\mathbb{R}^d} \ell^d(dy)\overline{f(y)}h(y) = \langle f, h\rangle. \qquad (6.1.3)\end{aligned}$$

Furthermore, it holds $\mathbf{v}_t\mathbf{v}_{-t} = \mathbf{v}_{-t}\mathbf{v}_t = \mathbb{1}$. $\square$

We define some more mappings:

$$\begin{aligned} s_t : \mathbb{R}^d \longrightarrow \mathbb{R}^d\ , &\qquad s_t(x) := e^t \cdot x &\qquad (x \in \mathbb{R}^d), &\qquad (6.1.4)\\ \sigma_t : M \longrightarrow M\ , &\qquad \sigma_t(\varphi) := \sum_{x \in supp\,\varphi} \varphi(\{x\})\delta_{e^t x} &\qquad (\varphi \in M), &\qquad (6.1.5)\\ S_t : \mathcal{M} \longrightarrow \mathcal{M}\ , &\qquad S_t\Psi := \Psi \circ \sigma_t &\qquad (\Psi \in \mathcal{M}). &\qquad (6.1.6)\end{aligned}$$

Lemma 6.2. *For all $t \in \mathbb{R}$ the second quantization $\Gamma(\mathbf{v}_t) \in \mathfrak{L}(\mathcal{M})$ of $\mathbf{v}_t$ is given by*

$$\Gamma(\mathbf{v}_t) = O_{(\mathbb{e}_{e^{\frac{td}{2}}})}S_t.$$

On exponential vectors $\mathbb{e}_h$ with $h \in \mathcal{L}^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\Gamma(\mathbf{v}_t)\mathbb{e}_h &= \mathbb{e}_{\mathbf{v}_t h} = \mathbb{e}_{e^{\frac{td}{2}}\cdot h\circ s_t} &\qquad (6.1.7)\\ \Gamma(\mathbf{v}_t)^*\mathbb{e}_h &= \mathbb{e}_{\mathbf{v}_{-t} h} = \mathbb{e}_{e^{-\frac{td}{2}}\cdot h\circ s_{-t}}. &\qquad (6.1.8)\end{aligned}$$

PROOF. Equations (6.1.8) and (6.1.7) follow directly from Definition 2.14, Remark 2.15 and Lemma 6.1. Using (2.3.4) in Lemma 2.13 we get for all $\mathbb{e}_h \in \mathcal{M}$, $\varphi \in M$

$$\begin{aligned}\Gamma(\mathbf{v}_t)\mathbb{e}_h(\varphi) &= \mathbb{e}_{\mathbf{v}_t h}(\varphi) = \mathbb{e}_{e^{\frac{td}{2}}}(\varphi)\cdot \mathbb{e}_{h\circ s_t}(\varphi) = \mathbb{e}_{e^{\frac{td}{2}}}(\varphi)\cdot \prod_{x\in\varphi}(h\circ s_t(x))^{\varphi(\{x\})}\\ &= \mathbb{e}_{e^{\frac{td}{2}}}(\varphi)\cdot \prod_{e^t x\in\sigma_t(\varphi)}(h(e^t x))^{\sigma_t(\varphi)(\{e^t x\})} = O_{\mathbb{e}_{e^{\frac{td}{2}}}}\mathbb{e}_h\circ\sigma_t(\varphi) = O_{\mathbb{e}_{e^{\frac{td}{2}}}}(S_t\mathbb{e}_h)(\varphi).\end{aligned}$$

The fact that the exponential vectors are total in $\mathcal{M}$ completes the proof. $\square$

Let for $g_1,\, g_2 \in \mathcal{L}^2(\mathbb{R}^d)$ the operator B_{g_1,g_2} be defined as it was given in (4.3.9). For exponential vectors with $h \in \mathcal{L}^2(\mathbb{R}^d)$ and $\varphi \in M$ we have

$$B_{g_1,g_2}\mathbb{e}_h(\varphi) = e^{\langle g_1, h\rangle}\cdot \mathbb{e}_{g_2}(\varphi). \qquad (6.1.9)$$

Lemma 6.3. *Let $\mathbf{v} : \mathcal{L}^2(\mathbb{R}^d) \longrightarrow \mathcal{L}^2(\mathbb{R}^d)$ be an isometry and h, g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$. Then*

$$e^{\langle g_1, \mathbf{v}(h)\rangle} \cdot \mathbb{e}_{\mathbf{v}^*(g_2)} = B_{\mathbf{v}^*(g_1),\mathbf{v}^*(g_2)} \mathbb{e}_h. \tag{6.1.10}$$

PROOF. From (6.1.9) there follows immediately for $\varphi \in M$

$$B_{\mathbf{v}^*(g_1),\mathbf{v}^*(g_2)} \mathbb{e}_h(\varphi) = e^{\langle \mathbf{v}^*(g_1), h\rangle} \cdot \mathbb{e}_{\mathbf{v}^*(g_2)}(\varphi) = e^{\langle g_1, \mathbf{v}(h)\rangle} \cdot \mathbb{e}_{\mathbf{v}^*(g_2)}(\varphi).$$

□

Proposition 6.4. *Let $t \in \mathbb{R}$ and g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$. Then it holds*

$$\Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \Gamma(\mathbf{v}_t) = B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)}. \tag{6.1.11}$$

PROOF. Using (6.1.9), (2.3.4) from Lemma 2.13 and Lemma 6.3 we get for $h \in \mathcal{L}^2(\mathbb{R}^d)$

$$\begin{aligned} \Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \Gamma(\mathbf{v}_t) \mathbb{e}_h &= \Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \mathbb{e}_{\mathbf{v}_t(h)} = \Gamma(\mathbf{v}_t)^* e^{\langle g_1, \mathbf{v}_t(h)\rangle} \mathbb{e}_{g_2} \\ &= e^{\langle g_1, \mathbf{v}_t(h)\rangle} \Gamma(\mathbf{v}_{-t}) \mathbb{e}_{g_2} = e^{\langle g_1, \mathbf{v}_t(h)\rangle} \mathbb{e}_{\mathbf{v}_{-t}(g_2)} = B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)} \mathbb{e}_h. \end{aligned}$$

□

6.2 Contractions and Invariant States

As we have seen $\Lambda_{Q,\beta}$ and $\Lambda_{M,\alpha}$ leave the local algebras invariant. Now we will consider Λ_{Q,β,U_2} and $\Lambda_{M,\alpha,U_1,U_2}$ with $U_i = \Gamma(\mathbf{v}_{t_i})$. The expansion (for $t_i > 0$) or contraction (for $t_i < 0$) will destroy the invariance.
For Borel sets $K \subseteq \mathbb{R}^d$ we identify again $h \in \mathcal{L}^2(K)$ with $h \in \mathcal{L}^2(\mathbb{R}^d)$ where $h(x) = 0$ for all $x \in K^c$.
The following Proposition is the analogue of Proposition 5.3 from[26].

Proposition 6.5. *For $K \in \mathfrak{B}$ and g_1, $g_2 \in \mathcal{L}^2(K)$, $t \in \mathbb{R}$ we have*

$$\Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \Gamma(\mathbf{v}_t) O_{M_{K^t}} \in \mathfrak{L}(\mathcal{M}_{K^t})$$

with $K^t := s_t(K) = \{e^t x \ ; \ x \in K\}$.

PROOF. From Proposition 6.4 and Lemma 6.3 follows for all $h \in \mathfrak{L}(\mathcal{M}_{K^t})$

$$\Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \Gamma(\mathbf{v}_t) \mathbb{e}_h = B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)} \mathbb{e}_h = e^{\langle \mathbf{v}_{-t}(g_1), h\rangle} \cdot \mathbb{e}_{\mathbf{v}_{-t}(g_2)}.$$

Moreover, we have $K = s_{-t}(K^t)$. So, for $g_j \in \mathcal{L}^2(K)$, $j = 1, 2$, holds $g_j \circ s_{-t} \in \mathcal{L}^2(K^t)$ and hence

$$\mathbf{v}_{-t}(g_j) = e^{-\frac{td}{2}} g_j \circ s_{-t} \in \mathcal{L}^2(K^t).$$

This shows that $\Gamma(\mathbf{v}_t)^* B_{g_1,g_2} \Gamma(\mathbf{v}_t)$ restricted to functions from $\mathcal{M}_{K^t}$ belongs to $\mathcal{L}^2(K^t)$. □

If we denote by $\mathcal{J}_{K^t}$ the embedding into the local algebra given by (5.1.2), with Proposition 6.5 we proved that

$$\mathcal{J}_{K^t}(\Gamma(\mathbf{v}_t)^* B_{g_1,g_2}\Gamma(\mathbf{v}_t)O_{M_{K^t}}) \in \mathcal{C}_{K^t}.$$

Now we will show that by the inner evolution described by the contraction the algebra $\mathcal{C}_K$ is mapped into $\mathcal{C}_{K^t}$. The corresponding result with a different representation of the contraction is Proposition 5.4 in [26].

Proposition 6.6. *For arbitrary $t \in \mathbb{R}$, $K \in \mathfrak{B}$ and $A \in \mathfrak{L}(\mathcal{M}_K)$ it holds*

$$\mathcal{J}_{K^t}(\Gamma(\mathbf{v}_t)^* A\Gamma(\mathbf{v}_t)O_{M_{K^t}}) = \Gamma(\mathbf{v}_t)^*\mathcal{J}_K(A)\Gamma(\mathbf{v}_t) \tag{6.2.1}$$

with $K^t = s_t(K)$.

PROOF. According to (5.1.3) for $K \in \mathfrak{B}$ and $A \in \mathfrak{L}(\mathcal{M}_K)$ on exponential vectors we have

$$\mathcal{J}_K(A) = \mathcal{S}^c(A \otimes O_{M_{K^c}})\mathcal{D}^c,$$

where in this case $K^c = \mathbb{R}^d \setminus K$.
For $h \in \mathcal{L}^2(\mathbb{R}^d)$, g_1, $g_2 \in \mathcal{L}^2(K)$ we get applying (2.3.12)

$$\begin{aligned} \mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h &= \mathcal{S}^c(B_{g_1,g_2} \otimes O_{M_{K^c}})\mathcal{D}^c\mathbb{e}_h = \mathcal{S}^c(B_{g_1,g_2}\mathbb{e}_h \otimes \mathbb{e}_{h_{|K^c}}) & (6.2.2)\\ &= \mathcal{S}^c(e^{\langle g_1,h\rangle}\mathbb{e}_{g_2} \otimes \mathbb{e}_{h_{|K^c}}) = e^{\langle g_1,h\rangle}\mathbb{e}_{g_2+h_{|K^c}}. & (6.2.3)\end{aligned}$$

Let $h \in \mathcal{L}^2(K^t)$ and g_1, $g_2 \in \mathcal{L}^2(K)$. From $h \in \mathcal{L}^2(K^t)$ follows $\mathbf{v}_t(h) \in \mathcal{L}^2(K)$. Consequently, we get applying Lemma 6.3 and Proposition 6.4

$$\begin{aligned} \Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)\mathbb{e}_h &= \Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_{\mathbf{v}_t(h)} = \Gamma(\mathbf{v}_t)^* e^{\langle g_1,\mathbf{v}_t(h)\rangle}\mathbb{e}_{g_2+\mathbf{v}_t(h_{|K^c})}\\ &= \Gamma(\mathbf{v}_t)^* e^{\langle g_1,\mathbf{v}_t(h)\rangle}\mathbb{e}_{g_2} = e^{\langle g_1,\mathbf{v}_t(h)\rangle}\mathbb{e}_{\mathbf{v}_{-t}(g_2)}\\ &= B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)}\mathbb{e}_h.\end{aligned}$$

We still have to show that $\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)$ restricted to $\mathcal{M}_{(K^t)^c}$ behaves like the identity. Let $h \in \mathcal{L}^2((K^t)^c)$. Because $x \in K^c$ is equivalent to $e^t x \in (K^t)^c$ we have $\langle g_1, \mathbf{v}_t(h)\rangle = 0$ for $g_1 \in \mathcal{L}^2(K)$, $\mathbf{v}_t(h_{|K^c}) = \mathbf{v}_t(h)$. Hence, for $h \in \mathcal{L}^2((K^t)^c)$, g_1, $g_2 \in \mathcal{L}^2(K)$

$$\begin{aligned} \Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)\mathbb{e}_h &= \Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_{\mathbf{v}_t(h)} = \Gamma(\mathbf{v}_t)^* e^{\langle g_1,\mathbf{v}_t(h)\rangle}\mathbb{e}_{g_2+\mathbf{v}_t(h_{|K^c})}\\ &= \Gamma(\mathbf{v}_t)^*\mathbb{e}_{\mathbf{v}_t(h)} = \mathbb{e}_{\mathbf{v}_{-t}(\mathbf{v}_t(h))} = \mathbb{e}_h.\end{aligned}$$

Consequently,

$$O_{M_{(K^t)^c}}\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v})O_{M_{(K^t)^c}} = O_{M_{(K^t)^c}} = \mathbb{1}_{\mathcal{M}_{(K^t)^c}}.$$

From Lemma 4.18 follows that (6.2.1) holds for all $A \in \mathfrak{L}(\mathcal{M}_K)$. □

The following Proposition is the analogue of Proposition 5.5 in [26].

Proposition 6.7. *For $K \in \mathfrak{B}$, $t \in \mathbb{R}$ and $g_1,\ g_2 \in \mathcal{L}^2(K)$ there holds*

$$\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(B_{g_1,g_2})) = \mathcal{J}_{K^t}(O_{M_{K^t}}\Lambda_{Q,\beta}(B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)})O_{M_{K^t}}) \tag{6.2.4}$$

with $K^t = s_t(K)$.

PROOF. From Propositions 6.6 and 6.4 we get

$$\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t) \;=\; \mathcal{J}_{K^t}(\Gamma(\mathbf{v}_t)^*B_{g_1,g_2}\Gamma(\mathbf{v}_t)O_{M_{K^t}}) = \mathcal{J}_{K^t}(B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)}).$$

Because of $\mathbf{v}_{-t}(g_j) \in \mathcal{L}^2(K^t)$ there follows from

$$\mathcal{J}_{K^t}(B_{g_1,g_2})\mathbb{e}_h = e^{\langle g_1,h\rangle}\mathbb{e}_{g_2+h_{|(K^t)^c}}$$

(see also proof of Proposition 6.6) for $h \in \mathcal{L}^2(\mathbb{R}^d)$

$$\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)\mathbb{e}_h = e^{\langle \mathbf{v}_{-t}(g_1),h\rangle}\mathbb{e}_{\mathbf{v}_{-t}(g_2)+h_{|(K^t)^c}}.$$

Furthermore, we have for all $A \in \mathfrak{L}(\mathcal{M})$ and $\alpha(x) := \sqrt{1-|\beta(x)|^2} \quad (x \in \mathbb{R}^d)$

$$\begin{aligned}\Lambda_{Q,\beta}(A)\mathbb{e}_h &= \mathcal{V}^*_{\alpha,\beta}(\mathbb{1}\otimes A)\mathcal{V}_{\alpha,\beta}\ \mathbb{e}_h = \mathcal{V}^*_{\alpha,\beta}(\mathbb{1}\otimes A)(O_{\mathbb{e}_\alpha}\otimes O_{\mathbb{e}_\beta})\mathcal{D}^c\mathbb{e}_h\\ &= \mathcal{V}^*_{\alpha,\beta}(\mathbb{1}\otimes A)(\mathbb{e}_{\alpha h}\otimes \mathbb{e}_{\beta h}) = \mathcal{V}^*_{\alpha,\beta}(\mathbb{e}_{\alpha h}\otimes A\mathbb{e}_{\beta h})\\ &= \mathcal{S}^c(O_{\mathbb{e}_{\overline{\alpha}}}\otimes O_{\mathbb{e}_{\overline{\beta}}})(\mathbb{e}_{\alpha h}\otimes A\mathbb{e}_{\beta h}) = \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h\otimes O_{\mathbb{e}_{\overline{\beta}}}A\mathbb{e}_{\beta h}})\end{aligned} \tag{6.2.5}$$

and

$$\begin{aligned}\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(B_{g_1,g_2})\mathbb{e}_h &= \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes O_{\mathbb{e}_{\overline{\beta}}}\Gamma(\mathbf{v}_t)^*B_{g_1,g_2}\Gamma(\mathbf{v}_t)\mathbb{e}_{\beta h})\\ &= \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes O_{\mathbb{e}_{\overline{\beta}}}\Gamma(\mathbf{v}_t)^*e^{\langle g_1,\mathbf{v}_t(\beta h)\rangle}\mathbb{e}_{g_2}) = e^{\langle g_1,\mathbf{v}_t(\beta h)\rangle}\mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes O_{\mathbb{e}_{\overline{\beta}}}\mathbb{e}_{\mathbf{v}^*_t(g_2)})\\ &= e^{\langle g_1,\mathbf{v}_t(\beta h)\rangle}\mathbb{e}_{\overline{\beta}\mathbf{v}_{-t}(g_2)+|\alpha|^2h}.\end{aligned} \tag{6.2.6}$$

Finally, we get from (6.2.5) and (6.2.6) for all $h \in \mathcal{L}^2(\mathbb{R}^d)$

$$\begin{aligned}&\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(B_{g_1,g_2}))\mathbb{e}_h = \Lambda_{Q,\beta}\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)\mathbb{e}_h\\ &= \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes O_{\mathbb{e}_{\overline{\beta}}}\Gamma(\mathbf{v}_t)^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_t)\mathbb{e}_{\beta h})\\ &= \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes O_{\mathbb{e}_{\overline{\beta}}}\ e^{\langle \mathbf{v}_{-t}(g_1),\beta h\rangle}\mathbb{e}_{\mathbf{v}_{-t}(g_2)+(\beta h)_{|(K^t)^c}})\\ &= e^{\langle \mathbf{v}_{-t}(g_1),\beta h\rangle}\ \mathcal{S}^c(\mathbb{e}_{|\alpha|^2h}\otimes \mathbb{e}_{\overline{\beta}\mathbf{v}_{-t}(g_2)+\overline{\beta}(\beta h)_{|(K^t)^c}}) = e^{\langle \mathbf{v}_{-t}(g_1),\beta h\rangle}\mathbb{e}_{\overline{\beta}\mathbf{v}_{-t}(g_2)+|\beta|^2h_{|(K^t)^c}+|\alpha|^2h}\\ &= e^{\langle \mathbf{v}_{-t}(g_1),\beta h\rangle}\mathbb{e}_{\overline{\beta}\mathbf{v}_{-t}(g_2)+h_{|(K^t)^c}+(1-|\beta|^2)h)_{|K^t}}\\ &= e^{\langle g_1,\mathbf{v}_t(\beta h)\rangle}\mathbb{e}_{\overline{\beta}\mathbf{v}_{-t}(g_2)+(1-|\beta|^2)h_{|K^t}+h_{|(K^t)^c}} = \mathcal{J}_{K^t}(O_{M_{K^t}}\Lambda_{Q,\beta}(B_{\mathbf{v}_{-t}(g_1),\mathbf{v}_{-t}(g_2)})O_{M_{K^t}})\mathbb{e}_h.\end{aligned}$$

□

Lemma 6.8. *For $K \in \mathfrak{B}$, $t \in \mathbb{R}$, $h \in \mathcal{L}^2(\mathbb{R}^d)$ and $g_1,\ g_2 \in \mathcal{L}^2(K)$ it holds*

$$\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(B_{g_1,g_2}))\mathbb{e}_h = e^{\langle g_1,\mathbf{v}_t(\beta h)\rangle}\mathbb{e}_{(1-|\beta|^2)h_{|K^t}+\overline{\beta}\mathbf{v}_{-t}(g_2)+h_{|(K^t)^c}} \tag{6.2.7}$$

with $K^t = s_t(K)$.

PROOF. This formula is obtained in the proof of Proposition 6.7.

□

Observe that for all $n \in \mathbb{N}$ there holds $(\Gamma(\mathbf{v}_t))^n = \Gamma(\mathbf{v}_t^n) = \Gamma(\mathbf{v}_{n\cdot t})$.
$\Lambda^n_{Q,\beta,U}$ can be calculated in a simple way for $\Lambda^n_{Q,\beta,U} = \Lambda_{Q,\beta^n,U^n}$ ([26], Prop. 3.28). For $U = \Gamma(\mathbf{v})$ this holds if and only if $\beta \cdot \mathbf{v}(h) = \mathbf{v}(\beta h)\ \forall\ h \in \mathcal{L}^2(G,\nu)$ (see [26], Prop. 3.29). For our contraction mapping $\mathbf{v}_t$ this means

$$[\beta \cdot (\mathbf{v}_t(h))](x) = \beta(x) \cdot e^{\frac{td}{2}} \cdot h(e^t x) = e^{\frac{td}{2}} \cdot (\beta h)(e^t x) = [\mathbf{v}_t(\beta h)](x)$$

for all $x \in \mathbb{R}^d$. This is the case if $\beta = \beta \circ s_t$. If this is required for all $t \in \mathbb{R}$, the function β has to be constant on all sets $\{e^t \cdot x\ ;\ t \in \mathbb{R}\}$ for arbitrary $x \in \mathbb{R}^d$. The most simple case where this condition is fulfilled is β being constant.
For the following proposition compare Proposition 5.7 in [26]. We will give a more detailed proof.

Proposition 6.9. *Let $t \in \mathbb{R}$, $\beta \in \mathbb{C}$ with $|\beta| \in [0,1]$. Then for all h, g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$ and $n \geq 1$ it holds*

$$\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}(B_{g_1,g_2})\mathbb{e}_h = e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n h\rangle}\ \mathbb{e}_{(1-|\beta|^{2n})h+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)}. \tag{6.2.8}$$

Furthermore, for all $K \in \mathfrak{B}$, g_1, $g_2 \in \mathcal{L}^2(K)$, $n \geq 1$, $h \in \mathcal{L}^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h &= e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n h\rangle} \cdot \mathbb{e}_{(1-|\beta|^{2n})h_{|K^{nt}}+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)+h_{|(K^{nt})^c}} \\ &= \mathcal{J}_{K^{nt}}(\widetilde{A})\mathbb{e}_h \end{aligned} \tag{6.2.9}$$

where $K^{nt} := s_{nt}(K) = \{e^{ntx}\ ;\ x \in K\}$, and $\widetilde{A} \in \mathfrak{L}(\mathcal{M}_{K^{nt}})$ is given for $f \in \mathcal{L}^2(K^{nt})$ by

$$\widetilde{A}\ \mathbb{e}_f = e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n f\rangle}\ \mathbb{e}_{(1-|\beta|^{2n})f+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)}. \tag{6.2.10}$$

PROOF. Let h, g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$. Using Lemma 6.8 we get

$$\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}(B_{g_1,g_2})\mathbb{e}_h = \Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}(B_{g_1,g_2})\mathbb{e}_h = e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n h\rangle} \cdot \mathbb{e}_{(1-|\beta|^{2n})h+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)}.$$

Now let $K \in \mathfrak{B}$, g_1, $g_2 \in \mathcal{L}^2(K)$, $h \in \mathcal{L}^2(\mathbb{R}^d)$. Then using Prop. 5.5 from [26] (with our contraction $\mathbf{v}_t$) we have

$$\begin{aligned}\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h &= \Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h \\ &= \mathcal{J}_{K^{nt}}(O_{M_{K^{nt}}}\Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}(B_{\mathbf{v}_{-nt}(g_1),\mathbf{v}_{-nt}(g_2)})O_{M_{K^{nt}}})\mathbb{e}_h.\end{aligned}$$

According to Lemma 6.8 there holds

$$\Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h = e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n h\rangle} \cdot \mathbb{e}_{(1-|\beta|^{2n})h_{|K^{nt}}+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)+h_{|(K^{nt})^c}},$$

or for $h \in \mathcal{L}^2(K^{nt})$

$$\Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h = e^{\langle \mathbf{v}_{-nt}(g_1),\beta^n h\rangle}\ \mathbb{e}_{(1-|\beta|^{2n})h+\overline{\beta}^n\mathbf{v}_{-nt}(g_2)}.$$

□

Now we will look for coherent locally normal states on the quasilocal algebra that are invariant under $\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}$ (compare Prop. 5.8 from [26]).

Proposition 6.10. *Let $\mathbf{\Phi}^h$ be a locally normal coherent state, $\beta \in \mathbb{C}$ a constant with $|\beta| \in [0,1]$ and let $t \in \mathbb{R}$. For all $K \in \mathfrak{B}$ and g_1, $g_2 \in \mathcal{L}^2(K)$ it holds*

$$\begin{aligned} \mathbf{\Phi}^h \quad &\circ \quad \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} \mathcal{J}_K(B_{g_1,g_2}) \\ &= \quad exp\{\langle \mathbf{v}_{-nt}(g_1), \beta^n h\rangle + \langle \beta^n h, \mathbf{v}_{-nt}(g_2)\rangle - |\beta|^{2n} \|h\|^2_{\mathcal{L}^2(K^{nt})}\}, \quad (6.2.11) \end{aligned}$$

where again $K^{nt} := s_{nt}(K)$.

PROOF. We know that among the normal states only the vacuum state is invariant or asymptotically invariant with respect to $\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}$.
Now let $\mathbf{\Phi}^h$ be a locally normal coherent state on the quasilocal algebra $\mathcal{C}$. Hence, $h : G \longrightarrow \mathbb{C}$ is locally square integrable with respect to the Lebesgue measure, i.e. $\|h\|^2_K := \|h\|^2_{\mathcal{L}^2(K)} = \int\limits_K |h(x)|^2 dx < \infty \ \forall\ K \in \mathfrak{B}$ and

$$\mathbf{\Phi}^h(\mathcal{J}_K(A)) = e^{-\|h\|_K} \langle \mathbb{e}_h, A\mathbb{e}_h\rangle_{\mathcal{M}_K} \qquad (K \in \mathfrak{B}\ ,\ A \in \mathfrak{L}(\mathcal{M}_K)).$$

from this and Proposition 6.9 follows

$$\begin{aligned} \mathbf{\Phi}^h \quad &\circ \quad \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(B_{g_1,g_2})) = \mathbf{\Phi}^h(\mathcal{J}_{K^{nt}}(\widetilde{A})) \\ &= \quad e^{-\|h\|_{K^{nt}}} \langle \mathbb{e}_h, \widetilde{A}\mathbb{e}_h\rangle_{\mathcal{M}_{K^{nt}}} \qquad (6.2.12) \end{aligned}$$

with $\widetilde{A}$ being the operator defined by (6.2.10).
So, we can continue (6.2.12):

$$\begin{aligned} &= exp\{-\|h\|^2_{K^{nt}}\} \langle \mathbb{e}_h, e^{\langle \mathbf{v}_{-nt}(g_1), \beta^n h\rangle} \cdot \mathbb{e}_{(1-|\beta|^{2n})h + \overline{\beta}^n \mathbf{v}_{-nt}(g_2)}\rangle_{\mathcal{M}_{K^{nt}}} \\ &= exp\{-\|h\|^2_{K^{nt}} + \langle \mathbf{v}_{-nt}(g_1), \beta^n h\rangle + \langle h, (1-|\beta|^{2n})h\rangle_{\mathcal{L}^2(K^{nt})} + \langle h, \overline{\beta}^n \mathbf{v}_{-nt}(g_2)\rangle\} \\ &= exp\{-\|h\|^2_{K^{nt}} + \langle \mathbf{v}_{-nt}(g_1), \beta^n h\rangle + (1-|\beta|^{2n})\|h\|^2_{K^{nt}} + \langle h, \overline{\beta}^n \mathbf{v}_{-nt}(g_2)\rangle\} \\ &= exp\{-|\beta|^{2n})\|h\|^2_{K^{nt}} + \langle \mathbf{v}_{-nt}(g_1), \beta^n h\rangle + \langle h, \overline{\beta}^n \mathbf{v}_{-nt}(g_2)\rangle\}. \end{aligned}$$

□

Now the question is, for which of these states $\mathbf{\Phi}^h$ we have invariance?

Proposition 6.11. *Let $\mathbf{\Phi}^h$ be a locally normal coherent state, $\beta \in \mathbb{C}$ a constant with $|\beta| \in [0,1]$ and let $t \in \mathbb{R}$.*
If h fulfills the condition

$$\frac{h(x)}{h(e^t x)} = \beta e^{\frac{td}{2}} \qquad f.a.a.\ x \in \mathbb{R}^d \qquad (6.2.13)$$

then $\mathbf{\Phi}^h$ is invariant with respect to $\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}$.

PROOF. For an arbitrary locally normal coherent state $\mathbf{\Phi}^h$ it holds

$$\begin{aligned}\mathbf{\Phi}^h(\mathcal{J}_K(B_{g_1,g_2})) &= e^{-\|h\|_K^2}\langle \mathbb{e}_h, B_{g_1,g_2}\mathbb{e}_h\rangle_{\mathcal{M}_K} = e^{-\|h\|_K^2}\langle \mathbb{e}_h, e^{\langle g_1,h\rangle}\cdot \mathbb{e}_{g_2}\rangle \\ &= exp\{\langle g_1,h\rangle + \langle h,g_2\rangle - \|h\|_K^2\}\end{aligned}$$

with $K \in \mathfrak{B}$ and $g_1,\ g_2 \in \mathcal{L}^2(K)$.
If we compare this with the result of Proposition 6.9 we have equality (for $n=1$) if and only if

$$\langle g_1,h\rangle + \langle h,g_2\rangle - \|h\|_K^2 = \langle g_1, \mathbf{v}_t(\beta h)\rangle + \langle \mathbf{v}_t(\beta h), g_2\rangle - |\beta|^2\|h\|_{K^t}^2. \tag{6.2.14}$$

Since $K^t = s_t(K) = \{e^t x\ ;\ x \in K\}$ we have $\|h\|_{K^t}^2 = e^{td}\|h\circ s_t\|_K^2$. Furthermore, it holds $\mathbf{v}_t(\beta h) = \beta\cdot\mathbf{v}_t(h) = \beta\cdot e^{\frac{td}{2}}\cdot(h\circ s_t)$. Hence, (6.2.14) holds if and only if

$$\begin{aligned}&\langle g_1,h\rangle + \langle h,g_2\rangle - \|h\|_K^2 \\ &\qquad = \langle g_1, \beta\cdot e^{\frac{td}{2}}(h\circ s_t)\rangle + \langle \beta\cdot e^{\frac{td}{2}}(h\circ s_t), g_2\rangle - e^{td}|\beta|^2\|h\circ s_t\|_K^2.\end{aligned}$$

This equality is true if and only if

$$h = \beta e^{\frac{td}{2}} h\circ s_t. \tag{6.2.15}$$

□

The simplest solution of (6.2.15) is

1. $\beta = e^{-\frac{td}{2}}$ and $h = h\circ s_t$, this may be obtained by constant h.

Other possible solutions of (6.2.15) are

2. $h(x) = ax^\gamma$ and $\beta = e^{-(\frac{td}{2}+\gamma t)}$ with constants a and γ such that $t(d+2\gamma) > 0$, in this case we have for all $x \in \mathbb{R}^d$
   $\beta e^{\frac{td}{2}} h\circ s_t(x) = \beta e^{\frac{td}{2}} h(e^t x) = \beta e^{\frac{td}{2}}(e^t x)^m = \beta e^{\frac{td}{2}} e^{tm}x^m = h(x)$.

3. $h(x) = a^{lnx}$ and $\beta = a^{-t}\cdot e^{-\frac{td}{2}}$ ($d=1$, $a=$ const. such that $|\beta| \in (0,1)$), then we have for all $x > 0$
   $\beta e^{\frac{td}{2}} h\circ s_t(x) = \beta e^{\frac{td}{2}} a^{ln(e^t x)} = \beta e^{\frac{td}{2}}\cdot a^{t+lnx} = \beta e^{\frac{td}{2}}\cdot a^t\cdot a^{lnx} = a^{lnx} = h(x)$.

For all possible solutions the contraction constant t has to be chosen such that $|\beta| < 1$.

To describe the evolution of the measurement apparatus we repeat the considerations for $\Lambda_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}$ defined in (4.3.7).
Remember that $\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})} = \Lambda^{n-1}_{Q,\sqrt{1-|\alpha|^2},\Gamma(\mathbf{v}_{t_2})} \circ \Lambda_{M,\alpha,\Gamma(\mathbf{v}_{t_1})}$ and $(\Gamma(\mathbf{v}_t))^n = \Gamma(\mathbf{v}_t^n) = \Gamma(\mathbf{v}_{nt})$.
The following Proposition corresponds to Proposition 6.9.

Proposition 6.12. *Let $t \in \mathbb{R}$, α a complex constant with $|\alpha| \in [0,1]$, $\beta := \sqrt{1-|\alpha|^2}$. Then for all h, g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$ and $n \geq 1$ it holds*

$$\begin{aligned}\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}&(B_{g_1,g_2})\mathbb{e}_h \\ &= e^{\langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha\beta^{n-1}h)\rangle} \cdot \mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})h+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)}. \end{aligned} \tag{6.2.16}$$

Furthermore, for all $K \in \mathfrak{B}$, g_1, $g_2 \in \mathcal{L}^2(K)$, $n \geq 1$, $h \in \mathcal{L}^2(\mathbb{R}^d)$ we get

$$\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h \tag{6.2.17}$$
$$= e^{\langle g_1, \mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta^{n-1}h)\rangle} \cdot \mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})h_{|K^{t_1+(n-1)t_2}}+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)+h_{|(K^{t_1+(n-1)t_2})^c}}$$
$$= \mathcal{J}_{K^{t_1+(n-1)t_2}}(\widetilde{A})\mathbb{e}_h \tag{6.2.18}$$

where $K^{t_1+(n-1)t_2} := s_{t_1+(n-1)t_2}(K) = \{e^{(t_1+(n-1)t_2)x} \; ; \; x \in K\}$, *and* $\widetilde{A} \in \mathfrak{L}(\mathcal{M}_{K^{t_1+(n-1)t_2}})$ *is given for* $f \in \mathcal{L}^2(K^{t_1+(n-1)t_2})$ *by*

$$\widetilde{A}\ \mathbb{e}_f = e^{\langle g_1, \alpha\beta^{n-1}\mathbf{v}_{t_1+(n-1)t_2}f\rangle}\ \mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})f+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)}. \tag{6.2.19}$$

PROOF. Let $\beta_1 := \beta^{n-1}$ and $\alpha_1 := \sqrt{1-|\beta_1|^2}$. Using Proposition 6.9 we get for h, g_1, $g_2 \in \mathcal{L}^2(\mathbb{R}^d)$

$$\begin{aligned}
\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}&(B_{g_1,g_2})\mathbb{e}_h = (\Lambda_{Q,\beta_1,\Gamma(\mathbf{v}_{(n-1)t_2})} \circ \Lambda_{M,\alpha,\Gamma(\mathbf{v}_{t_1})})(B_{g_1,g_2})\mathbb{e}_h \\
&= (\Lambda_{Q,\beta_1,\Gamma(\mathbf{v}_{(n-1)t_2})}(\mathcal{V}^*_{\alpha,\beta}(\Gamma(\mathbf{v}_{t_1})^* B_{g_1,g_2}\Gamma(\mathbf{v}_{t_1}) \otimes \mathbb{1})\mathcal{V}_{\alpha,\beta}))\mathbb{e}_h \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{1} \otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(\Gamma(\mathbf{v}_{-t_1})B_{g_1,g_2}\Gamma(\mathbf{v}_{t_1}) \otimes \mathbb{1})\mathcal{V}_{\alpha,\beta}\Gamma(\mathbf{v}_{(n-1)t}))\mathcal{V}_{\alpha_1,\beta_1}\mathbb{e}_h \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(\Gamma(\mathbf{v}_{-t_1})B_{g_1,g_2}\Gamma(\mathbf{v}_{t_1}) \otimes \mathbb{1})\mathcal{V}_{\alpha,\beta}\mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\beta_1 h)}) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(B_{\mathbf{v}_{-t_1}g_1,\mathbf{v}_{-t_1}g_2}(\mathbb{e}_{\mathbf{v}_{(n-1)t}(\alpha\beta_1 h)} \otimes \mathbb{e}_{\mathbf{v}_{(n-1)t}(\beta^n h)}))) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(O_{e^{\langle \mathbf{v}_{-t_1}g_1, \mathbf{v}_{(n-1)t_2}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-t_1}g_2} \otimes \mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\beta^n h)})) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes \Gamma(\mathbf{v}_{-(n-1)t_2})O_{e^{\langle g_1, \mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\overline{\alpha}\mathbf{v}_{-t_1}g_2+|\beta|^2\mathbf{v}_{(n-1)t_2}(\beta_1 h)}) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes O_{e^{\langle g_1, \mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-(n-1)t_2}(\overline{\alpha}\mathbf{v}_{-t_1}g_2+|\beta|^2\mathbf{v}_{(n-1)t_2}(\beta_1 h))}) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \otimes O_{e^{\langle g_1, \mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-(n-1)t_2-t_1}(\overline{\alpha}g_2)+|\beta|^2\beta_1 h}) \\
&= e^{\langle g_1, \mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta^{n-1}h)\rangle} \cdot \mathbb{e}_{|\alpha_1|^2 h+\overline{\beta_1}(\mathbf{v}_{-(n-1)t_2-t_1}(\overline{\alpha}g_2)+|\beta|^2\beta_1 h)} \\
&= e^{\langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha\beta^{n-1}h)\rangle} \cdot \mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})h+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)}.
\end{aligned}$$

Furthermore, we get from this and Proposition 6.6 for g_1, $g_2 \in \mathcal{L}^2(K)$

$$\begin{aligned}
\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}&\mathcal{J}_K(B_{g_1,g_2})\mathbb{e}_h = \Lambda_{Q,\beta_1,\Gamma(\mathbf{v}_{n-1)t_2})} \circ \Lambda_{M,\alpha,\Gamma(\mathbf{v}_{t_1})}(\mathcal{J}_K(B_{g_1,g_2}))\mathbb{e}_h \\
= &\ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \\
&\otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(\Gamma(\mathbf{v}_{t_1})^*\mathcal{J}_K(B_{g_1,g_2})\Gamma(\mathbf{v}_{t_1})(\mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\alpha\beta_1 h)} \otimes \mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\beta^n h)}))) \\
= &\ \mathcal{V}^*_{\alpha_1,\beta_1}(\mathbb{e}_{\alpha_1 h} \\
&\otimes \Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}(\mathcal{J}_{K^{t_1}}(B_{\mathbf{v}_{-t_1}g_1,\mathbf{v}_{-t_1}g_2}O_{M_{K^{t_1}}})(\mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\alpha\beta_1 h)} \otimes \mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\beta^n h)}))
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{V}^*_{\alpha_1,\beta_1}\big(\mathbb{e}_{\alpha_1 h} \\
&\otimes\Gamma(\mathbf{v}_{-(n-1)t_2})\mathcal{V}^*_{\alpha,\beta}\big(O_{e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-t_1}g_2+\mathbf{v}_{(n-1)t_2}(\alpha\beta_1 h)_{|_{(K^{t_1+(n-1)t_2})^c}}}\otimes\mathbb{e}_{\mathbf{v}_{(n-1)t_2}(\beta^n h)}\big)\big) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}\big(\mathbb{e}_{\alpha_1 h} \\
&\otimes\Gamma(\mathbf{v}_{-(n-1)t})O_{e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\overline{\alpha}(\mathbf{v}_{-t_1}g_2+\mathbf{v}_{(n-1)t_2}(\alpha\beta_1 h)_{|_{(K^{t_1+(n-1)t_2})^c}})+|\beta|^2\mathbf{v}_{(n-1)t_2}(\beta_1 h)}\big) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}\big(\mathbb{e}_{\alpha_1 h} \\
&\otimes O_{e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-(n-1)t_2}(\overline{\alpha}\mathbf{v}_{-t_1}g_2+\mathbf{v}_{(n-1)t_2}(|\alpha|^2\beta_1 h)_{|_{(K^{t_1+(n-1)t_2})^c}})+|\beta|^2\mathbf{v}_{(n-1)t_2}(\beta_1 h))}\big) \\
&= \mathcal{V}^*_{\alpha_1,\beta_1}\big(\mathbb{e}_{\alpha_1 h}\otimes O_{e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta_1 h)\rangle}}\mathbb{e}_{\mathbf{v}_{-(n-1)t_2-t_1}(\overline{\alpha}g_2)+|\alpha|^2\beta^{n-1}h_{|_{(K^{t_1+(n-1)t_2})^c}})+|\beta|^2\beta_1 h}\big) \\
&= e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta^{n-1}h)\rangle}\cdot\mathbb{e}_{|\alpha_1|^2 h+\overline{\alpha\beta_1}(\mathbf{v}_{-(n-1)t_2-t_1}(g_2)+|\alpha|^2|\beta|^{2(n-1)}h_{|_{(K^{t_1+(n-1)t_2})^c}})+|\beta|^{2n}h)} \\
&= e^{\langle g_1,\mathbf{v}_{(n-1)t_2+t_1}(\alpha\beta^{n-1}h)\rangle}\cdot \\
&\qquad\qquad\cdot\mathbb{e}_{(1-|\alpha|^2|\beta|^{2(n-1)})h_{|K^{t_1+(n-1)t_2}}+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)+h_{|(K^{t_1+(n-1)t_2})^c}} \\
&= \mathcal{J}_{K^{t_1+(n-1)t_2}}(\widetilde{A})\mathbb{e}_h.
\end{aligned}
$$

□

The following Proposition is analogue to Proposition 6.10.

Proposition 6.13. *Let $\mathbf{\Phi}^h$ be a locally normal coherent state, α a complex constant satisfying $|\alpha|\in[0,1]$. Let $t\in\mathbb{R}$ and $\beta:=\sqrt{1-|\alpha|^2}$. For all $K\in\mathfrak{B}$ and g_1, $g_2\in\mathcal{L}^2(K)$ holds*

$$
\begin{aligned}
\mathbf{\Phi}^h\circ\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}\mathcal{J}_K(B_{g_1,g_2}) = exp\{&\langle\mathbf{v}_{-t_1-(n-1)t_2}(g_1),\alpha\beta^{n-1}h\rangle \\
&+\langle\alpha\beta^{n-1}h,\mathbf{v}_{-t_1-(n-1)t_2}(g_2)\rangle-|\alpha|^2|\beta|^{2(n-1)}\|h\|^2_{K^{t_1+(n-1)t_2}}\},
\end{aligned}
\quad (6.2.20)
$$

where again $K^{t_1+(n-1)t_2}:=s_{t_1+(n-1)t_2}(K)$ and $\|h\|_{K^{t_1+(n-1)t_2}}:=\|h\|_{\mathcal{L}^2(K^{t_1+(n-1)t_2})}$.

PROOF. Let $\mathbf{\Phi}^h$ be a locally normal coherent state on the quasilocal algebra $\mathcal{C}$. Hence, $h:G\longrightarrow\mathbb{C}$ is locally square integrable with respect to the Lebesgue measure, i.e. $\|h\|^2_K=\int\limits_K|h(x)|^2dx<\infty\ \forall\ K\in\mathfrak{B}$ and

$$
\mathbf{\Phi}^h(\mathcal{J}_K(A)) = e^{-\|h\|^2_{\mathcal{L}^2(K)}}\langle\mathbb{e}_h,A\mathbb{e}_h\rangle_{\mathcal{M}_K}\qquad(K\in\mathfrak{B}\ ,\ A\in\mathfrak{L}(\mathcal{M}_K)).
$$

from this and Proposition 6.12 follows

$$
\begin{aligned}
\mathbf{\Phi}^h\quad\circ\quad&\Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}(\mathcal{J}_K(B_{g_1,g_2})) = \mathbf{\Phi}^h(\mathcal{J}_{K^{t_1+(n-1)t_2}}(\widetilde{A}) \\
&= e^{-\|h\|^2_{K^{t_1+(n-1)t_2}}}\langle\mathbb{e}_h,\widetilde{A}\mathbb{e}_h\rangle_{\mathcal{M}_{K^{t_1+(n-1)t_2}}}
\end{aligned}
\quad (6.2.21)
$$

with $\widetilde{A}$ being the operator defined by (6.2.19).

So, we can continue (6.2.21):

$$
\begin{aligned}
&= exp\{-\|h\|^2_{K^{t_1+(n-1)t_2}}\} \cdot \\
&\quad \cdot \ \langle e_h, e^{\langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha\beta^{n-1}h\rangle} \cdot e_{(1-|\alpha\beta^{n-1}|^2)h+\overline{\alpha|\beta|^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)}\rangle_{\mathcal{M}_{K^{t_1+(n-1)t_2}}} \\
&= exp\{-\|h\|^2_{K^{t_1+(n-1)t_2}} \\
&\quad + \langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha\beta^{n-1})h\rangle \cdot \langle e_h, e_{(1-|\alpha\beta^{n-1}|^2)h+\overline{\alpha\beta^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)}\rangle_{\mathcal{M}_{K^{t_1+(n-1)t_2}}}\} \\
&= exp\{-\|h\|^2_{K^{t_1+(n-1)t_2}} + \langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha|\beta|^{n-1}h)\rangle \\
&\quad + \langle h, (1-|\alpha\beta^{n-1}|^2)h\rangle_{\mathcal{L}^2(K^{t_1+(n-1)t_2})} + \langle h, \overline{\alpha|\beta|^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)\rangle_{\mathcal{L}^2(K^{t_1+(n-1)t_2})}\} \\
&= exp\{-\|h\|^2_{K^{t_1+(n-1)t_2}} + (1-|\alpha\beta^{n-1}|^2)\|h\|^2_{K^{t_1+(n-1)t_2}} \\
&\quad + \langle g_1, \mathbf{v}_{t_1+(n-1)t_2}(\alpha|\beta|^{n-1}h)\rangle + \langle h, \overline{\alpha|\beta|^{n-1}}\mathbf{v}_{-t_1-(n-1)t_2}(g_2)\rangle\} \\
&= exp\{-|\alpha\beta^{n-1}|^2)\|h\|^2_{K^{t_1+(n-1)t_2}} + \langle \mathbf{v}_{-t_1-(n-1)t_2}(g_1), \alpha\beta^{n-1}h\rangle \\
&\quad + \langle \alpha\beta^{n-1}h, \mathbf{v}_{-t_1-(n-1)t_2}(g_2)\rangle\}.
\end{aligned}
$$

□

6.3 Convergence to Invariant States

As we have seen in section 6.1 for suitable choice of the parameters α, β and t we find states that are invariant under $\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}$. Now we want to search for conditions on the state ω that ensure convergence of $\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}$ to such an invariant non-vacuum state.
First we give some examples for convergence to invariant states.

Example 6.1. *Let $R > 0$ and denote by $B_R(0)$ the ball of radius R around the origin. Consider the function that is 0 inside and equal to $a \in \mathbb{C}$ outside the ball:*

$$h(x) := a \cdot \chi_{(B_R(0))^c}(x). \tag{6.3.1}$$

For $h \in \mathcal{L}^2(\mathbb{R}^d)$ we know from Proposition 6.10 that $\mathbf{\Phi}^h \circ \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} = \mathbf{\Phi}^{\beta^n \mathbf{v}_{nt}(h)}$. We choose $\beta = e^{-\frac{td}{2}}$. Then equation (6.2.15) implies

$$\beta^n \mathbf{v}_{nt} h(x) = \beta^n e^{\frac{ntd}{2}} \cdot h(e^{nt} \cdot x) = a \cdot \chi_{(B_R(0))^c}(e^{nt} \cdot x) = a \cdot \chi_{(B_{e^{-nt}}(0))^c}(x).$$

For $t > 0$ this converges to $a \cdot \chi_{\mathbb{R}^d}(x)$ as n tends to infinity. Hence we have

$$\mathbf{\Phi}^h \circ \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{} \mathbf{\Phi}^a. \tag{6.3.2}$$

Of course this works also for $\Lambda_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})}$.
For $h \in \mathcal{L}^2(\mathbb{R}^d)$ we know from Proposition 6.12 that $\mathbf{\Phi}^h \circ \Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})} = \mathbf{\Phi}^{\alpha\beta^{n-1}\mathbf{v}_{t_1+(n-1)t_2}h}$. We choose $\alpha = e^{\frac{-t_1 d}{2}}$ and $\beta = e^{\frac{-t_2 d}{2}}$ with t_1, $t_2 > 0$ such that $|\alpha|^2 + |\beta|^2 = 1$. Then equation (6.2.15)implies

$$
\begin{aligned}
\alpha\beta^{n-1}\mathbf{v}_{t_1+(n-1)t_2}h(x) &= \alpha\beta^{n-1}e^{\frac{t_1 d}{2}+\frac{(n-1)t_2 d}{2}} \cdot h(e^{t_1+(n-1)t_2} \cdot x) \\
&= a \cdot \chi_{(B_R(0))^c}(e^{t_1+(n-1)t_2} \cdot x) = a \cdot \chi_{(B_{e^{-t_1-(n-1)t_2}(0)})^c}(x).
\end{aligned}
$$

For $t_2 > 0$ this converges to $a \cdot \chi_{\mathbb{R}^d}(x)$ as n tends to infinity. Hence we also have

$$\mathbf{\Phi}^h \circ \Lambda^n_{M,\alpha,\Gamma(\mathbf{v}_{t_1}),\Gamma(\mathbf{v}_{t_2})} \xrightarrow[n\to\infty]{} \mathbf{\Phi}^a. \tag{6.3.3}$$

We do the same for another function.

Example 6.2. *For $a \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ with $t(d+2\gamma) > 0$ consider the function that is 0 inside and equal to $a \cdot x^\gamma$ for x from outside the ball:*

$$h(x) := a \cdot x^\gamma \cdot \chi_{(B_R(0))^c}(x). \tag{6.3.4}$$

We choose $\beta = e^{-(\frac{td}{2}+\gamma t)}$. Then equation (6.2.15) implies

$$\begin{aligned} \beta^n \mathbf{v}_{nt} h(x) &= \beta^n e^{\frac{ntd}{2}} \cdot h(e^{nt} \cdot x) = e^{-n(\frac{td}{2}+\gamma t)} e^{\frac{ntd}{2}} a \cdot (e^{nt} x)^\gamma \cdot \chi_{(B_R(0))^c}(e^{nt} \cdot x) \\ &= a x^\gamma \cdot \chi_{(B_R(0))^c}(e^{nt} \cdot x) = a x^\gamma \cdot \chi_{(B_{e^{-nt}}(0))^c}(x). \end{aligned}$$

For $t > 0$ this converges to $a \cdot x^\gamma \cdot \chi_{\mathbb{R}^d}(x)$ as n tends to infinity. Hence we have

$$\mathbf{\Phi}^h \circ \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{} \mathbf{\Phi}^g. \tag{6.3.5}$$

with $g(x) := a \cdot x^\gamma$.

In Proposition 5.3 we extended the concept of position distribution to locally normal states.
Now in this context we cite the definition of the reduced and conditional reduced density matrix from [26].

Definition 6.14. *Let ω be a locally normal state. Then ω possesses the CONDITIONAL REDUCED DENSITY MATRIX $k_\omega : M^3 \longrightarrow \mathbb{C}$ if for all $K \in \mathfrak{B}$ all $\mathfrak{M}$-measurable functions f vanishing outside M_K and all Hilbert Schmidt operators A on $\mathcal{M}_K$ with kernel $l : M^2 \longrightarrow \mathbb{C}$ such that $\mathcal{S}^c(O_f \otimes A)\mathcal{D}^c$ extends to a bounded operator on $\mathcal{M}_K$ the formula*

$$\omega(\mathcal{J}_K(\mathcal{S}^c(O_f \otimes A)\mathcal{D}^c)) = \int Q_\omega(d\varphi) f(\varphi_K) \int F_K^2(d[\varphi_1,\varphi_2]) l(\varphi_1,\varphi_2) k_\omega(\varphi_2,\varphi_1,\varphi)$$

holds true.

Definition 6.14 shows that ω is determined by Q_ω and k_ω and vice versa. States for which a conditional reduced density matrix in the sense of Definition 6.14 exists are called Σ'-states. These states can be characterized by a property of the position distribution alone. For more details see [30].

Definition 6.15. *Let ω be a locally normal state which possesses a conditional reduced density matrix k_ω and let the function $k_\omega(\varphi_1,\varphi_2,\cdot)$ be Q_ω-integrable for almost all φ_1, φ_2.*
Then the function $r_\omega : M^2 \longrightarrow \mathbb{C}$ given by

$$r_\omega(\varphi_1,\varphi_2) = \int Q_\omega(d\varphi) k_\omega(\varphi_1,\varphi_2,\varphi) \tag{6.3.6}$$

is called REDUCED DENSITY MATRIX ([7]).

If the reduced density matrix exists it does not determine the state as the conditional reduced density matrix does.
Now we will first look for conditions such that $\omega \circ \Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}$ converges weak-$*$ly to a coherent state $\mathbf{\Phi}^a$ with a constant $a \in \mathbb{C}$. We will need some preparations.

Lemma 6.16. *Let $K \in \mathfrak{B}$ and A a Hilbert Schmidt operator on $\mathcal{M}_K$ with kernel l. Furthermore, let $\Gamma(\mathbf{v}_t)$ be the second quantization of the contraction operator $\mathbf{v}_t$ defined in 6.1.1.*
Then $\widehat{A} := O_{\overline{\beta}}\Gamma(\mathbf{v}^*)A\Gamma(\mathbf{v})O_\beta$ is a Hilbert Schmidt operator on $\mathcal{M}_{K^t}$ with kernel

$$\widehat{l}(\varphi_1, \varphi_2) = \mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\varphi_1) \cdot \mathbb{e}_{e^{-\frac{td}{2}}\beta}(\varphi_2) \cdot l(\sigma_{-t}(\varphi_1), \sigma_{-t}(\varphi_2)). \tag{6.3.7}$$

PROOF. For $h \in \mathcal{L}^2(K)$ we have

$$\begin{aligned} A\Gamma(\mathbf{v}_t)O_\beta \mathbb{e}_h(\varphi) &= A\Gamma(\mathbf{v}_t)\mathbb{e}_{\beta h}(\varphi) = A\mathbb{e}_{\mathbf{v}_t(\beta h)}(\varphi) = A\mathbb{e}_{e^{\frac{td}{2}}\cdot\beta h}(\sigma_t\varphi) \\ &= \int F_K(d\varphi_1) l(\varphi, \varphi_1)\mathbb{e}_{e^{\frac{td}{2}}\cdot\beta h}(\sigma_t\varphi_1) \end{aligned} \tag{6.3.8}$$

with σ_t defined in (6.1.5). From Lemma 6.2 we know that $\Gamma(\mathbf{v}_t^*) = O_{\mathbb{e}_{e^{-\frac{td}{2}}}} S_{-t}$ with S_{-t} defined in (6.1.6). From this, (6.3.8) and the transformation rule

$$\int F_K(d\varphi)h(\sigma_t(\varphi)) = \int F_{K^t}(d\varphi)\mathbb{e}_{e^{-td}}(\varphi)h(\varphi) \tag{6.3.9}$$

we conclude

$$\begin{aligned} O_{\overline{\beta}}\Gamma(\mathbf{v}_t^*)A\Gamma(\mathbf{v}_t)O_\beta\mathbb{e}_h(\varphi) &= O_{\mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}} S_{-t}A\Gamma(\mathbf{v}_t)O_\beta\mathbb{e}_h(\varphi) \\ &= \mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\varphi) \int F_K(d\varphi_1) l(\sigma_{-t}(\varphi), \varphi_1)\mathbb{e}_{e^{\frac{td}{2}}\cdot\beta h}(\sigma_t(\varphi_1)) \\ &= \int F_K(d\varphi_1)\mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\varphi)\mathbb{e}_{e^{\frac{td}{2}}\cdot\beta}(\sigma_t(\varphi_1))\mathbb{e}_h(\sigma_t(\varphi_1)) l(\sigma_{-t}(\varphi), \varphi_1) \\ &= \int F_{K^t}(d\varphi_1)\mathbb{e}_{e^{-td}}(\varphi_1)\mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\varphi)\mathbb{e}_{e^{\frac{td}{2}}\beta}(\varphi_1)\mathbb{e}_h(\varphi_1) l(\sigma_{-t}(\varphi), \sigma_{-t}(\varphi_1)) \\ &= \int F_{K^t}(d\varphi_1)\mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\varphi)\mathbb{e}_{e^{-\frac{td}{2}}\beta}(\varphi_1) l(\sigma_{-t}(\varphi), \sigma_{-t}(\varphi_1))\mathbb{e}_h(\varphi_1) \\ &= \int F_{K^t}(d\varphi_1)\widehat{l}(\varphi, \varphi_1)\mathbb{e}_h(\varphi_1). \end{aligned}$$

□

Lemma 6.17. *Let $K \in \mathfrak{B}$ and A a Hilbert Schmidt operator on $\mathcal{M}_K$ with kernel l. Furthermore, let $\Gamma(\mathbf{v}_t)$ be the second quantization of the contraction operator $\mathbf{v}_t$ defined in 6.1.1 and ω a locally normal state with conditional reduced density matrix k_ω. Then there holds*

$$\begin{aligned} \omega(\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A))) &= \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^2}(\varphi_{K^t}) \\ &\int F_K^2(d[\varphi_1, \varphi_2])\mathbb{e}_{e^{\frac{td}{2}}\overline{\beta}\circ s_t}(\varphi_1)\mathbb{e}_{e^{\frac{td}{2}}\beta\circ s_t}(\varphi_2) l(\varphi_1, \varphi_2) k_\omega(\sigma_t(\varphi_2), \sigma_t(\varphi_1), \varphi). \end{aligned} \tag{6.3.10}$$

PROOF. From Proposition 6.6, Lemma 6.16 and Proposition 5.3 we get

$$\omega(\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A)) = \omega(\mathcal{J}_{K^t}(\Lambda_{Q,\beta,\Gamma(\mathbf{v}_t)}(A))$$
$$= \omega(\mathcal{J}_{K^t}(\mathcal{S}^c(O_{|\alpha|^2} \otimes O_{\overline{\beta}}\Gamma(\mathbf{v}_t^*)A\Gamma(\mathbf{v}_t)O_\beta)\mathcal{D}^c) = \omega(\mathcal{J}_{K^t}(\mathcal{S}^c(O_{|\alpha|^2} \otimes \widehat{A})\mathcal{D}^c)$$
$$= \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^2}(\varphi_{K^t})$$
$$\int F^2_{K^t}d([\varphi_1,\varphi_2])\widehat{l}(\varphi_1,\varphi_2)k_\omega(\varphi_2,\varphi_1,\varphi)\mathbb{e}_{e^{-td}}(\varphi_1)\mathbb{e}_{e^{-td}}(\varphi_2)\mathbb{e}_{e^{td}}(\varphi_1)\mathbb{e}_{e^{td}}(\varphi_2)$$
$$= \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^2}(\varphi_{K^t})$$
$$\int F^2_K d([\varphi_1,\varphi_2])\mathbb{e}_{e^{td}}(\sigma_t(\varphi_1))\mathbb{e}_{e^{td}}(\sigma_t(\varphi_2))\widehat{l}(\sigma_t(\varphi_1),\sigma_t(\varphi_2))k_\omega(\sigma_t(\varphi_2),\sigma_t(\varphi_1),\varphi)$$
$$= \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^2}(\varphi_{K^t}) \int F^2_K d([\varphi_1,\varphi_2])\mathbb{e}_{e^{td}}(\sigma_t(\varphi_1))\mathbb{e}_{e^{td}}(\sigma_t(\varphi_2))$$
$$\mathbb{e}_{e^{-\frac{td}{2}}\overline{\beta}}(\sigma_t(\varphi_1))\mathbb{e}_{e^{-\frac{td}{2}}\beta}(\sigma_t(\varphi_2))l(\varphi_1,\varphi_2)k_\omega(\sigma_t(\varphi_2),\sigma_t(\varphi_1),\varphi)$$
$$= \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^2}(\varphi_{K^t})$$
$$\int F^2_K d([\varphi_1,\varphi_2])\mathbb{e}_{e^{\frac{td}{2}}\overline{\beta}\circ s_t}(\varphi_1)\mathbb{e}_{e^{\frac{td}{2}}\beta\circ s_t}(\varphi_2)l(\varphi_1,\varphi_2)k_\omega(\sigma_t(\varphi_2),\sigma_t(\varphi_1),\varphi).$$

□

Now we consider the case $\beta = e^{-\frac{td}{2}}$ with a contraction constant $t > 0$.
We will give a necessary und sufficient condition for convergence of $\mathbf{\Phi}^a$.
The following proposition containing a different representation of the contraction operator can be found in [26] as Prop. 5.14. We want to give a more detailed proof.

Proposition 6.18. *Let $K \in \mathfrak{B}$, $\Gamma(\mathbf{v}_t)$ the second quantization of the contraction operator $\mathbf{v}_t$ defined in 6.1.1 and ω a locally normal state with conditional reduced density matrix k_ω.*
Then it holds $\omega \circ \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^a$ if and only if the condition

(K) For $n \longrightarrow \infty$ the function

$$(\varphi_1,\varphi_2) \mapsto \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^{2n}}(\varphi_{K^{nt}})k_\omega(\sigma_{nt}(\varphi_2),\sigma_{nt}(\varphi_1),\varphi)$$

converges for every $K \in \mathfrak{B}$ weakly in $\mathcal{M}_K \otimes \mathcal{M}_K$ to the function

$$(\varphi_1,\varphi_2) \mapsto e^{-|a|^2\cdot\ell^d(K)} \cdot \mathbb{e}_{\overline{a}}(\varphi_1)\mathbb{e}_a(\varphi_2).$$

is fulfilled.

PROOF. Let (K) be valid. Because β is constant we know from Lemma 4.20 that $\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} = \Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}$. We apply Lemma 6.17 and obtain for each $K \in \mathfrak{B}$ and Hilbert Schmidt operators A on $\mathcal{M}_K$ with kernel l

$$\omega(\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A))) \xrightarrow[n\to\infty]{} \int F^2_K(d[\varphi_1,\varphi_2]) l(\varphi_1,\varphi_2) e^{-|a|^2 \ell^d(K)} \cdot \mathbb{e}_{\overline{a}}(\varphi_1)\mathbb{e}_a(\varphi_2) = \Phi^a(\mathcal{J}_K(A)). \tag{6.3.11}$$

Because all operators B_{g_1,g_2} defined by (4.3.9) are Hilbert Schmidt operators they fulfil (6.3.11). This and Lemma 5.12 imply $\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^a$.
Now let $\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^a$. Hence,

$$\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A)) \xrightarrow[n\to\infty]{} \Phi^a(\mathcal{J}_K(A))$$

for all $K \in \mathfrak{B}$ and all Hilbert Schmidt operators A on $\mathcal{M}_K$. If A has kernel l then

$$\Phi^a(\mathcal{J}_K(A)) = e^{-\|a\|^2_K} \int F^2_K(d[\varphi_1,\varphi_2]) l(\varphi_1,\varphi_2) \mathbb{e}_{\overline{a}}(\varphi_1)\mathbb{e}_a(\varphi_2).$$

This implies (K). □

Now we give sufficient conditions such that (K) is valid.
The analogue of the following proposition is Prop. 5.15 in [26]. We will give a different, more detailed proof.

Proposition 6.19. *Suppose that ω is a locally normal state such that*

$(K'1)$ For Q_ω-a.a. $\varphi \in M$ there holds

$$\frac{\varphi([-n,n)^d)}{(2n)^d} \xrightarrow[n\to\infty]{} |a|^2, \tag{6.3.12}$$

$(K'2)$ For F^2-a.a. (φ_1,φ_2) the reduced density matrix r_ω of ω fulfills

$$r_\omega(\sigma_{nt}(\varphi_1), \sigma_{nt}(\varphi_2)) \xrightarrow[n\to\infty]{} \mathbb{e}_{\overline{a}}(\varphi_1)\mathbb{e}_a(\varphi_2), \tag{6.3.13}$$

$(K'3)$ $|k_\omega(\varphi_1,\varphi_2,\varphi)| \le c^{\varphi_1(\mathbb{R}^d)+\varphi_2(\mathbb{R}^d)+1}$ uniformly in φ_1, φ_2 for some $c > 0$.

Then

$$\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^a.$$

PROOF. Fix $K = [-m, m)^d$. Then we have $K^{nt} = s_{nt}(K) = [-e^{nt}m, e^{nt}m)^d$. $(K'1)$ implies

$$\frac{\varphi(K^{nt})}{\ell^d(K^{nt})} \xrightarrow[n\to\infty]{} |a|^2.$$

Hence,

$$\mathbb{e}_{1-|\beta|^{2n}}(\varphi_{K^{nt}}) = \mathbb{e}_{1-e^{-ntd}}(\varphi_{K^{nt}}) = (1-e^{-ntd})^{\varphi(K^{nt})} = (1-e^{-ntd})^{\ell^d(K^{nt})\cdot\frac{\varphi(K^{nt})}{\ell^d(K^{nt})}}$$
$$\xrightarrow[n\to\infty]{} e^{-(2m)^d\cdot|a|^2}.$$

Moreover, for square integrable l on $M_K \times M_K$ we get using $(K'2)$

$$\int Q_\omega(d\varphi) l(\varphi_1, \varphi_2) k_\omega(\sigma_{nt}(\varphi_2), \sigma_{nt}(\varphi_1), \varphi) = l(\varphi_1, \varphi_2) r_\omega(\sigma_{nt}(\varphi_2), \sigma_{nt}(\varphi_1), \varphi)$$
$$\xrightarrow[n\to\infty]{} l(\varphi_1, \varphi_2)\mathbb{e}_{\overline{a}}(\varphi_1)\mathbb{e}_a(\varphi_2).$$

With $(K'3)$ holds

$$|l(\varphi_1, \varphi_2) k_\omega(\sigma_{nt}(\varphi_2), \sigma_{nt}(\varphi_1), \varphi)| \le c^{\varphi_1(\mathbb{R}^d)+\varphi_2(\mathbb{R}^d)+1}|l(\varphi_1, \varphi_2)|,$$

and $c^{\varphi_1(\mathbb{R}^d)+\varphi_2(\mathbb{R}^d)}$ is a $(F_K)^2$ integrable function. Thus we can apply the dominated convergence theorem and get (K).

□

Remark 6.20. *$(K'1)$ in Proposition 6.19 in an individual ergodic theorem for Q_ω. According to [25], section 3.2, the limit $s(\varphi) = \lim\limits_{n\to\infty} \frac{\varphi([-n,n)^d)}{(2n)^d}$ represents the individual intensity of φ. $|a|^2$ represents the intensity of the position distribution of the limit state $\mathbf{\Phi}^a$. If Q_ω is stationary the limit $s(\varphi)$ exists in any case but may be non-constant([25], Theorem 3.2.1). But if Q_ω is ergodic with finite intensity, the limit is constant Q_ω-a.s. over whole M ([25], section 6.10).*

Now we will look for conditions for convergence to a coherent state $\mathbf{\Phi}^h$ where h is not necessarily constant. We will use the function considered in Example 6.2.

Proposition 6.21. *Let $K \in \mathfrak{B}$, $\Gamma(\mathbf{v}_t)$ the second quantization of the contraction operator $\mathbf{v}_t$ defined in 6.1.1 and ω a locally normal state with conditional reduced density matrix k_ω. Furthermore, let h be a function from $\mathcal{L}^2(K)$ satisfying* (6.2.13) *with corresponding constant $\beta \in \mathbb{C}$, $|\beta| \in (0, 1)$.*
Then it holds $\omega \circ \Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^h$ if and only if the condition

$(\widetilde{K})$ For $n \longrightarrow \infty$ the function

$$(\varphi_1, \varphi_2) \mapsto \int Q_\omega(d\varphi)\mathbb{e}_{1-|\beta|^{2n}}(\varphi_{K^{nt}}) k_\omega(\sigma_{nt}(\varphi_2), \sigma_{nt}(\varphi_1), \varphi)$$

converges for every $K \in \mathfrak{B}$ weakly in $\mathcal{M}_K \otimes \mathcal{M}_K$ to the function

$$(\varphi_1, \varphi_2) \mapsto e^{-\|h\|^2_K} \cdot \mathbb{e}_{\overline{h}}(\varphi_1)\mathbb{e}_h(\varphi_2).$$

is fulfilled.

PROOF. Let $(\widetilde{K})$ be valid. Because β is constant we know from Lemma 4.20 that $\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)} = \Lambda_{Q,\beta^n,\Gamma(\mathbf{v}_{nt})}$. We apply Lemma 6.17 and obtain for each $K \in \mathfrak{B}$ and Hilbert Schmidt operators A on $\mathcal{M}_K$ with kernel l

$$\omega(\Lambda^n_{Q,\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A))) \xrightarrow[n\to\infty]{} \int F_K^2(d[\varphi_1,\varphi_2])l(\varphi_1,\varphi_2)e^{-\|h\|_K^2} \cdot e_{\overline{h}}(\varphi_1)e_h(\varphi_2) = \Phi^h(\mathcal{J}_K(A)). \tag{6.3.14}$$

Because all operators B_{g_1,g_2} defined by (4.3.9) are Hilbert Schmidt operators they fulfil (6.3.14). This and Lemma 5.12 imply $\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^h$.

Now let $\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)} \xrightarrow[n\to\infty]{weak-*} \Phi^h$. Hence,

$$\omega \circ \Lambda^n_{Q\beta,\Gamma(\mathbf{v}_t)}(\mathcal{J}_K(A)) \xrightarrow[n\to\infty]{} \Phi^h(\mathcal{J}_K(A))$$

for all $K \in \mathfrak{B}$ and all Hilbert Schmidt operators A on $\mathcal{M}_K$. If A has kernel l then

$$\Phi^h(\mathcal{J}_K(A)) = e^{-\|h\|_K^2} \int F_K^2(d[\varphi_1,\varphi_2])l(\varphi_1,\varphi_2)e_{\overline{h}}(\varphi_1)e_h(\varphi_2).$$

This implies $(\widetilde{K})$.

$\square$

Remark 6.22. *It seems natural to look for sufficient conditions (analogue to those in Proposition 6.19) for $(\widetilde{K})$ to be valid. If we use a condition analogue to $(K'1)$, to show convergence for instance to $\mathbf{\Phi}^h$ with a function $h(x) = a \cdot x^\gamma$ (see Example 6.2), we get (compare to the proof of Proposition 6.19)*

$$\begin{aligned} e_{1-|\beta|^{2n}}(\varphi_{K^{nt}}) &= e_{(1-e^{-ntd-2\gamma nt})}(\varphi_{K^{nt}}) = (1-e^{-nt(d+2\gamma)})^{\varphi(K^{nt})} \\ &= (1-e^{-nt(d+2\gamma)})^{e^{ntd}\cdot(2m)^d\cdot\frac{\varphi(K^{nt})}{\ell^d(K^{nt})}} \end{aligned}$$

which converges to zero as n tends to infinity.
So we have to notice that it is not possible to give an analogue to Proposition 6.19 in such a simple way.

Bibliography

[1] Luigi Accardi. Noncommutative Markov chains associated to a preassigned evolution: an application to the quantum theory of measurement. *Adv. in Math.*, 29(2):226–243, 1978.

[2] L. Accardi and M. Ohya. Compound channels, transition expectations, and liftings. *Appl Math Optim*, 39:33–59, 1999.

[3] L. Accardi and G. S. Watson. Quantum random walks. In L. Accardi and W. von Waldenfels, editors, *Quantum probabitity and Applications IV*, volume 1396 of *Lecture Notes in Mathematics*, pages 73–88. Springer-Verlag Berlin, Heidelberg, New York, 1989.

[4] L. Accardi. An Introduction to Quantum Markov Chains. preprint 498, Centro Vito Volterra, December 2001.

[5] H. Bauer. *Mass- und Integrationstheorie.* De Gruyter Lehrbuch. de Gruyter, Berlin, New York, 1990.

[6] H. Bauer. *Wahrscheinlichkeitstheorie.* De Gruyter Lehrbuch. de Gruyter, Berlin, New York, 4th edition, 1991.

[7] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, volume 2. Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[8] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, volume 1. Springer-Verlag, Berlin, Heidelberg, New York, 2nd edition, 1987.

[9] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Springer Series in Statistics. Springer-Verlag New York, Berlin, Heidelberg, 1988.

[10] T. Damaschke. *Zustände unendlicher Bosonensysteme auf dem Gitter.* PhD thesis, FSU Jena, 19.

[11] A.S. Dawydow. *Quantenmechanik.* Barth Verlagsgesellschaft, Leipzig, Berlin, Heidelberg, 8th edition, 1992.

[12] K.-H. Fichnter and G. Winkler. Generalized Brownian Motion, Point Processes and Stochastic Calculus for Random Fields. *Math. Nachr.*, 161:291–307, 1993.

[13] K.-H. Fichtner and W. Freudenberg. On a probabilistic model of infinite quantum mechanical particle systems. *Math. Nachr.*, 121:171–210, 1985.

[14] K.-H. Fichtner and W. Freudenberg. Point Processes and Normal States of Boson Systems. preprint, NTZ Leipzig, 1986.

[15] K.-H. Fichtner and W. Freudenberg. Point Processes and States of Infinite Boson Systems. preprint, NTZ Leipzig, 1986.

[16] K.-H. Fichtner and W. Freudenberg. Point processes and the position distribution of infinite boson systems. *J. Stat. Phys.*, 47:959–978, 1987.

[17] K.-H. Fichtner and W. Freudenberg. Characterization of states of infinite boson systems i. on the construction of states of boson systems. *Comm. Math. Phys.*, 137:315–357, 1991.

[18] K.-H. Fichtner and W. Freudenberg. Remarks on stochasic calculus on the Fock space. In L. Accardi, editor, *Quantum probabitity and Related Topics VII*, pages 305–323. World Scientific Publishing Co. Singapore, New Jersey, London, Hong Kong, 1991.

[19] K.-H. Fichtner, V. Liebscher, and W. Freudenberg. Time Evolution and Invariance of Boson Systems Given by Beam Splittings. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 1(4):511–531, 1998.

[20] K.-H. Fichtner and M. Ohya. Quantum Teleportation with Entangled States Given by Beam Splittings. *Comm. Math. Phys.*, 222:229–247, 2001.

[21] K.-H. Fichtner. Charakterisierung Poissonscher zufälliger Punktfolgen und infinitesimale Verdünnungsschemata. *Math. Nachr.*, 68:93–104, 1975.

[22] W. Freudenberg. Characterization of states of infinite boson systems ii. on the existence of the conditional reduced density matrix. *Comm. Math. Phys.*, 137:461–472, 1991.

[23] W. Freudenberg. Quantum Stochastic Integrals on a General Fock Space. *Quantum Probability and Related Topics*, VIII:189–210, 1993.

[24] W. Freudenberg. On a Class of Quantum Markov Chains on the Fock Space. *Quantum Probability and Related Topics*, IX:215–237, 1994.

[25] J. Kerstan, K. Matthes, and J. Mecke. *Unbegrenzt teilbare Punktprozesse.* Akademie-Verlag, Berlin, 1974.

[26] W. Freudenberg K.-H. Fichtner and V. Liebscher. Beam Splittings and Time Evolutions of Boson Systems. Forschungsergebnisse der Fakultät für Mathematik und Informatik Math/Inf/96/39, Friedrich-Schiller-Universität Jena, December 1996.

[27] W. Freudenberg K.-H. Fichtner and V. Liebscher. Nonindependent Splittings and Gibbs States. *Mathematical Notes*, 64(4):518–523, 1998.

[28] W. Freudenberg K.-H. Fichtner and V. Liebscher. Characterization of Classical and Quantum Poisson Systems by Thinnings and Splittings. *Math. Nachr.*, 218:25–47, 2000.

[29] W. Freudenberg K.-H. Fichtner and V. Liebscher. On Exchange Mechanisms for Bosons. *Infinite Dim. Anal., Quantum Prob. and rel. topics*, submitted.

[30] V. Liebscher. *Bedingte Zustände und Charakterisierung lokalnormaler Zustände von unendlichen Bosonensystemen.* PhD thesis, FSU Jena, 1993.

[31] V. Liebscher. Conditional local states and decomposition of locally normal states. *Math. Nachr.*, 164:103–117, 1993.

[32] V. Liebscher. *Beam Splittings, Coherent States und Quantum Stochastic Differential Equations.* Habilitation thesis, FSU Jena, 1998.

[33] V. Liebscher. A Limit Theorem for Quantum Markov Chains associated to Beam Splittings. *Open Systems and Information Dynamics*, 8:1–29, 2001.

[34] V. Liebscher. Isometric Cocycles related to Beam Splittings. In W. Freudenberg, editor, *Proceedings of the Conference "Quantum Probability and Infinite-dimensional Analysis", Burg, Germany 15 - 20 March 2001*, volume 15 of *QP - PQ Quantum probability and White Noise Analysis*, pages 161–180. World Scientific, Singapore, 2003.

[35] J. M. Lindsay and H. Maasen. An integral kernel approach to noise. In L. Accardi and W. von Waldenfels, editors, *Quantum probabitity and Applications III*, volume 1303 of *Lecture Notes in Mathematics*, pages 192–208. Springer-Verlag Berlin, Heidelberg, New York, 1988.

[36] P. A. Meyer. *Quantum Probability for Probabilists*, volume 1538 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[37] K. R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus.* Monographs in mathematics ; vol. 85. Birkhäuser Verlag, Basel, Boston, Berlin, 1992.

[38] B. E. A. Saleh R. A. Campos and M.C. Teich. Quantum-mechanical lossless beam splitter: SU(2) symmetry and photon statistics. *Phys. Rev. A*, 40(3):1371–1384, 1989.

[39] A.N. Shiryaev. *Probability.* Springer-Verlag, Berlin, Heidelberg, New York, 2nd edition, 1995.

List of Symbols

Algebras

$\mathcal{A}$	C^*-algebra, 9
$\mathcal{A}$	von Neumann algebra, 12
$\mathcal{B}$	C^*-algebra, 9
$\mathcal{B}$	von Neumann algebra, 13
$\mathcal{B}^{[k,n]}$	von Neumann algebra of observables from times k to n , 12
$\mathcal{B}_n$	von Neumann algebra, 13
$\mathcal{B}^n$	finite tensor product, 13
$\bigotimes_{\mathbb{N}} \mathcal{B}$	infinite tensor product, 13
$\mathcal{B} \otimes \mathcal{A}$	fixed C^*-tensor product, 9
$\mathcal{C}$	C^*-algebra, 9
$\mathcal{C}$	quasilocal algebra, 13, 75
$\mathcal{C} \otimes_{ql} \mathcal{C}$	tensor product for quasilocal algebras, 77
$\mathcal{C}_I$	local algebra, 13
$\mathcal{C}_K$	local algebra, 75
$\mathcal{C}_{n]}$	local algebra of the past up to time n, 14
$\mathcal{D}$	C^*-algebra, 9
$\mathcal{D}^{\otimes n}$	finite tensor product, 9
$\mathcal{D}^n$	finite tensor product, 9
Δ	function on $\mathbb{N}_0$, 52
$\mathfrak{L}(\mathcal{H})$	von Neumann algebra of all bounded linear operators over the Hilbert space $\mathcal{H}$, 9
$\mathcal{U}_j$	C^*-algebra, 12

Elements, Sets and Set Systems

$\emptyset$	empty set, 19
$\mathfrak{B}$	ring of all bounded sets in $\mathfrak{G}$, 16
$\mathbb{C}$	set of all complex numbers
$\mathfrak{G}$	BOREL σ-algebra on G, 16
J	family of finite subsets of $\mathbb{N}$, 13
K^t	set from $\mathfrak{B}$, 83
$\mathfrak{M}$	canonical σ-algebra over M, 16
$\mathfrak{M}_K$	restriction of $\mathfrak{M}$ to $K \in \mathfrak{B}$, 74

M^f set of all finite point configurations, 16
M^m set of all m-dimensional vectors with components from M, 16
M_n set of all n-point configurations, 16
$\mathbb{N}$ set of all natural numbers starting with 1
$\mathbb{N}_0$ set of all natural numbers including 0, 16
$n]$ past, set of all natural numbers from 1 to n, 10
φ locally finite counting measure, 16
$\widehat{\varphi}$ locally finite counting measure, 16
$\underline{\varphi}$ vector with components from M, 16
φ_K restriction of the configuration φ to $K \in \mathfrak{G}$, 74
$\mathbb{R}$ set of all real numbers
$\mathcal{S}(\mathcal{D})$ set of all states on $\mathcal{D}$, 9
Y set from $\mathfrak{M}$, 17
$\underline{Y}$ set from $\mathfrak{M}^n$, 31

Functions

$\langle .,. \rangle_{\mathcal{M}^n}$ scalar product in $\mathcal{M}^n$, 17
a kernel of the integral operator A, 44
α splitting rate, 56
β splitting rate, 56
χ_Y indicator function of the set Y, 17
$\mathbb{e}_h$ exponential vector, 25
f_K restriction of the function f to $K \in \mathfrak{B}$, 77
g splitting function, 32
g_n splitting function, 36
h_j bounded function, 38
h_n splitting function for independent beam splitting, 64
l kernel of an integral operator A, 92
$\widehat{l}$ kernel of an integral operator $\widehat{A}$, 93
ρ kernel of a density matrix K, 40
$\rho_n^{U_1,U_2}$ kernel of a density matrix of the state $\omega_n^{U_1,U_2}$, 43
$\rho_{n]}^{U_1,U_2}$ kernel of a density matrix $K_{n]}^{U_1,U_2}$, 41

Maps

$\|.\|$ norm,
$\|.\|_K$ norm in $\mathcal{L}^2(K)$, 87
j_i embedding, 13
$j_{[1,n]}$ embedding, 13
$\mathcal{J}_K$ embedding of $\mathfrak{L}(\mathcal{M}_K)$ in $\mathfrak{L}(\mathcal{M})$, 75

Measures

$\mathbb{B}_{m,p}$	binomial distribution with parameters m and p, 65
δ_x	Dirac measure on G, 16
δ_{δ_x}	Dirac measure on M, 66
F	σ-finite measure on $[M, \mathfrak{M}]$, Fock space measure, 17
F_K	restriction of F to $K \in \mathfrak{B}$, 74
$\mathbb{G}_p$	geometric distribution with parameter p, 54
$\mathbb{H}_{N,m,n}$	hypergeometric distribution with parameters N, m, n, 54
H_n	stochastic kernel, 50, 64
κ	Radon Nikodym derivative of Q_τ with respect to F, 47
Λ	intensity measure of a point process, 68
Λ_n	intensity measure of a point process, 68
λ	locally finite measure on $[G, \mathfrak{G}]$, 17
ν	locally finite measure on $[G, \mathfrak{G}]$, 16
$\mathfrak{o}$	zero measure in M, 16
Q_n	position distribution of the state ω_n, 48
$Q_{n]}$	position distribution of the state $\omega_{n]}$, 48
$Q_{n]}^{U_1,U_2}$	position distribution of the state $\omega_{n]}^{U_1,U_2}$, 48
$Q_n^{U_1,U_2}$	position distribution of the state $\omega_n^{U_1,U_2}$, 48
Q_τ	position distribution of the initial state τ, 47

Operators

$\mathbb{1}$	unit of a C^*-algebra, 9
$\mathbb{1}_{\mathcal{D}}$	unit in $\mathcal{D}$, 9
$\mathbb{1}_{\mathcal{D}}^n$	unit in $\mathcal{D}^{\otimes n}$, 9
$\mathbb{1}_{\mathcal{M}_K}$	identity in $\mathfrak{L}(\mathcal{M}_K)$, 75
$*$	convolution of measures, 26
A	integral operator,
$\widetilde{A}$	operator from $\mathfrak{L}(\mathcal{M}_{K^{nt}})$, 86
B_{g_1,g_2}	test operator, 69
$\mathcal{D}^c$	compound Malliavin derivative, 28
$(\mathcal{D}^c)^n$	n-fold compound Malliavin derivative, 30
$\mathcal{E}$	transition expectation, 10
$\mathcal{E}^*$	lifting, 9
$\mathcal{E}_{\alpha,\beta}$	transition expectation for independent beam splitting, 57
$\mathcal{E}_{\alpha,\beta,U_1,U_2}$	transition expectation for independent beam splitting, 57
$\mathcal{E}^{[k,n]}$	transition expectation, 12
$\mathcal{E}_n$	transition expectation, 12
$\mathcal{E}^{n]}$	transition expectation, 12
$\mathcal{E}_{U_1,U_2}$	transition expectation, 34
$\mathcal{E}_{U_1,U_2}^n$	transition expectation, 34
$\mathcal{F}_j$	transition expectation, 12

$\mathcal{V}_j$	isometric operator, 11
$\mathcal{V}_{U_1,U_2}$	isometric operator, 34
$\mathcal{V}^n_{U_1,U_2}$	isometric operator, 35

Spaces

G	complete separable metric space, phase space, 16
$\Gamma(\mathcal{H})$	symmetric Fock space over the Hilbert space $\mathcal{H}$, 17
$\mathcal{H}$	Hilbert space, 9
$\mathcal{L}^2(G,\nu)$	space of square integrable functions on G
$\mathcal{L}^2_{loc}(G,\nu)$	space of locally square integrable functions on G, 77
M	set of all locally finite counting measures on $[G,\mathfrak{G}]$, 16
M_K	restriction of M to $K \in \mathfrak{B}$, 74
$\mathcal{M}$	symmetric Fock space over G, 17
$\mathcal{M}_K$	symmetric Fock space over G restricted to $K \in \mathfrak{B}$, 74
$\mathcal{M}^n$	n-fold tensor product of $\mathcal{M}$, 17

States

$\emptyset^0$	vacuum state on $\mathfrak{L}(\mathcal{M})$, 70
$\emptyset^0_{\mathcal{A}}$	vacuum state on $\mathcal{A}$, 70
$\mathbf{\Phi}^h$	coherent state, 77
ω	state on a C^*-algebra, 9
ω_1	state of a quantum Markov chain at time 1, 13
$\omega_{[k,n]}$	state of a quantum Markov chain from time k to time n, 13
ω_n	state of a quantum Markov chain at time n, 46
$\omega_n^{U_1,U_2}$	state of a quantum Markov chain at time n, 39
$\omega_{n]}$	state of a quantum Markov chain up to time n, 14
$\omega_{n]}^{U_1,U_2}$	state of a quantum Markov chain up to time n, 39
ρ	state on a C^*-algebra, 10
τ	initial state of a quantum Markov chain, 14

Other Symbols

$\binom{n}{k}$	binomial coefficient, 18
$\binom{\varphi}{\widehat{\varphi}}$	generalized binomial coefficient, 18
a.a.	almost all
a.e.	almost everywhere
B^c	set theoretical complement of B, 18
$B_R(0)$	ball of radius R around the origin, 91
$\delta_{i,j}$	Kronecker delta symbol, 17
$\lvert\varphi\rvert$	number of points in the configuration φ, 16
$\varphi_{\vert B}$	restriction of the configuration φ to $B \in \mathfrak{B}$, 18

Lebenslauf

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Cottbus, den